x	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	0.9846	0.9412	0.8767	0.8	0.7191	0.64	0.5664	0.5

Then $I = \frac{0.125}{2}\left[1.5 + 2(0.9846 + 0.9412 + 0.8767 + 0.8 + 0.7191 + 0.64 + 0.5664)\right]$

$= 0.78475$

Thus we have $I_1 = 0.775$, $I_2 = 0.7828$

$I_3 = 0.78475$.

First Improved value b/n I_1 and I_2

$I = I_2 + \frac{I_2 - I_1}{3}$

$= 0.7828 + \frac{0.7828 - 0.775}{3}$

$= 0.7854$

First Improved value b/n I_2 and I_3

$I = I_3 + \frac{I_3 - I_2}{3}$

$= 0.78475 + \frac{0.78475 - 0.7828}{3}$

$= 0.7854$

Since these two values are identical,

By actual integration

$\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4} = 0.7854$

$\Rightarrow \pi = 3.1416$.

The following tables gives the velocity of a particle at time t

t(sec):	0	2	4	6	8	10	12
v(m/s):	4	6	16	34	60	94	136

Find the distance moved by the particle in 12 sec and also the acceleration at $t = 2$ sec.

Soln

We know $\frac{ds}{dt} = v \quad \frac{dv}{dt} = a$

$\therefore ds = v\,dt \Rightarrow s = \int v\,dt$

The distance moved by the particle in 12 seconds $= \int_0^{12} v\,dt$

Here $h = 2$, By Simpson's 1/3 rule

$$\therefore \int_0^{12} v\,dt = \frac{h}{3}\left[(v_0+v_6) + 4(v_1+v_3+v_5) + 2(v_2+v_4)\right]$$

$$= \frac{2}{3}\left[(4+136) + 4(6+34+94) + 2(16+90)\right]$$

$\therefore$ Distance covered $= 552$ m

Acceleration $\left.\frac{dv}{dt}\right|_{t=2}$

$$\therefore \left.\frac{dv}{dt}\right|_{t=2} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \cdots\right]$$

$$= \frac{1}{2}\left[10 - 8/2\right] = 3\ \text{m/sec}^2$$

Two Point Gaussian Quadrature Formula

$$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

Three Point Gaussian Quadrature Formula

$$\int_{-1}^{1} f(x)\,dx = \frac{8}{9} f(0) + \frac{5}{9}\left[f\left(-\sqrt{3/5}\right) + f\left(\sqrt{3/5}\right)\right]$$

Evaluate $I = \int_{-1}^{1} \frac{dx}{1+x^2}$ by two point and three point Gaussian Quadrature formula.

Soln By two point formula

$$\int_{-1}^{1} f(x)\,dx = f\left(-\sqrt{\tfrac{1}{3}}\right) + f\left(\sqrt{\tfrac{1}{3}}\right)$$

$$\therefore \int_{-1}^{1} \frac{1}{1+x^2}\,dx = \frac{1}{1+\frac{1}{3}} + \frac{1}{1+\frac{1}{3}}$$

$$= \frac{3}{4} + \frac{3}{4} = 1.5$$

By three point formula

$$\int_{-1}^{1} f(x)\,dx = \frac{8}{9} f(0) + 5/9\left[f\left(-\sqrt{3/5}\right) + f\left(\sqrt{3/5}\right)\right]$$

$$\therefore \int_{-1}^{1} \frac{1}{1+x^2}\,dx = \frac{8}{9}\left(\frac{1}{1+0}\right) + \frac{5}{9}\left[\frac{5}{8} + \frac{5}{8}\right]$$

$$= \frac{8}{9} + \frac{25}{36} = \frac{19}{12} = 1.5833.$$

But exact value

$$2\int_0^1 \frac{dx}{1+x^2} = 2\tan^{-1}(x)\Big|_0^1$$

$$= 2\,\pi/4 = \pi/2 = 1.5708.$$

Use Gaussian two point formula to Evaluate $\int_1^2 \frac{dx}{x}$

Soln

$a = 1 \quad b = 2$

The transformation is

$$x = \frac{1}{2}(b-a)t + \frac{1}{2}(b+a)$$

$$x = \frac{t+3}{2}$$

$$\int_1^2 \frac{dx}{x} = \int_{-1}^{1} \frac{2}{t+3}\left(\frac{dt}{2}\right) = \int_{-1}^{1} \frac{dt}{t+3}$$

By Gaussian two point formula

$$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{3-\frac{1}{\sqrt{3}}} + \frac{1}{3+\frac{1}{\sqrt{3}}}$$

$$= \sqrt{3}\left[\frac{1}{3\sqrt{3}-1} + \frac{1}{3\sqrt{3}+1}\right]$$

$$= \frac{18}{6} = \frac{9}{3} = 0.6923$$

Evaluate Double integrals By.

(i) Trapezoidal Rule

(ii) Simpson's Rule.

Evaluate $I = \int_0^1 \int_0^1 xe^y \, dx dy$ using the trapezoidal Rule and Simpson's Rule

Soln

Take $h = k = 0.5$

Table value of xe^y are given as follows.

		x_0	x_1	x_2
y \ x		0	0.5	1
y_0	0	0	0.5	1
y_1	0.5	0.5	0.82	1.65
y_2	1	0	1.36	2.72

By Trapezoidal rule

$$I = \int_0^1 \int_0^1 xe^y \, dx dy$$

$$I = \int_{y_0}^{y_2} \int_{x_0}^{x_2} f(x,y) \, dx dy$$

where $x_0 = 0$, $x_2 = 1$, $y_0 = 0$, $y_2 = 1$

$f(x,y) = xe^y$.

Evaluate $\int_{0.2}^{1.5} e^{-x^2} dx$ using Gaussian three point Quadrature formula

Soln:-

Let $x = \frac{(b-a)t + (b+a)}{2}$

Then $dx = \frac{b-a}{2} dt$

$\therefore x = \frac{(1.5-0.2)t + (1.5+0.2)}{2}$

$\Rightarrow x = 0.65t + 0.85$

$dx = 0.65\, dt$

$\therefore I = \int_{-1}^{1} e^{-(0.65t+0.85)^2} (0.65)\, dt$

$= 0.65 \left[0.5555 \exp\left[-((-0.7746)(0.65)+0.85)^2\right] \right.$

$+ (0.8889) \exp\left[-(0.85)^2\right]$

$\left. + 0.5555 \exp\left\{-\left[(0.7746)(0.65) + (0.85)\right]^2\right\} \right]$

$= 0.65 \left[0.49265 + 0.43159 + 0.08894 \right]$

$= 0.6585$

$$I = \frac{hk}{4}\Big[f(x_0, y_0) + 2f(x_0, y_1) + f(x_0, y_2) + 2f(x_1, y_0) + 4f(x_1, y_1) + 2f(x_1, y_2) + f(x_2, y_0) + 2f(x_2, y_1) + f(x_2, y_2)\Big]$$

$$= \frac{0.25}{4}\Big[0 + 2(0) + 2(0.5) + 4(0.82) + 2(1.36) + 1 + 2(1.65) + 2.72\Big]$$

$$= 0.876$$

(ii) By simpson's rule, we have

$$I = \frac{hk}{9}\Big[f(x_0, y_0) + 4f(x_0, y_1) + f(x_0, y_2) + 4f(x_1, y_0) + 16f(x_1, y_1) + 4f(x_1, y_2) + f(x_2, y_0) + 4f(x_2, y_1) + f(x_2, y_2)\Big]$$

$$= \frac{0.25}{9}\Big[0 + 4(0) + 4(0.5) + 16(0.82) + 4(1.36) + 1 + 4(1.65) + 2.72\Big]$$

$$= 0.858$$

Evaluate $\int_0^1 \int_0^1 e^{x+y}\,dx\,dy$ using the trapezoidal rule and simpson's rule.

Soln Take $h = k = 0.5$

The table values of e^{x+y} are given as follows.

y \ x	0	0.5	1
0	1	1.6487	2.7183
0.5	1.6487	2.7183	4.4817
1	2.7183	4.4817	7.3891

(i) using Trapezoidal Rule

$$I = \frac{hk}{4}\Big[f(x_0,y_0) + 2f(x_0,y_1) + f(x_0,y_2)$$
$$+ 2f(x_1,y_0) + 4f(x_1,y_1) + 2f(x_1,y_2)$$
$$+ f(x_2,y_0) + 2f(x_2,y_1) + f(x_2,y_2)\Big]$$

$$= \frac{1}{16}\Big[1 + 2(1.6487) + 2.7183 + 2(1.6487)$$
$$+ 4(2.7183) + 2(4.4817) + 2.7183$$
$$+ 2(4.4817) + 7.3891\Big]$$

$$= \frac{1}{16}\Big[1 + 4(1.6487) + 6(2.7183) + 4(4.4817)$$
$$+ 7.3891\Big]$$

$= 3.0762$.

(ii) using Simpson's Rule

$$I = \frac{hk}{9}\Big[f(x_0,y_0) + 4f(x_0,y_1) + f(x_0,y_2)$$
$$+ 4f(x_1,y_0) + 16f(x_1,y_1) + 4f(x_1,y_2)$$
$$+ f(x_2,y_0) + 4f(x_2,y_1) + f(x_2,y_2)\Big]$$

$$I = \frac{0.25}{9}\Big[1 + 4(1.6487) + 2.7183$$
$$+ 4(1.6487) + 16(2.7183) + 4(4.4817)$$
$$+ 2.7183 + 4(4.4817) + 7.3891\Big]$$
$$= \frac{26.59042}{9}$$
$$= 2.95430$$

Numerical Differentiation and Integration

Introduction:- Numerical differentiation

Numerical differentiation is the process of computing the value of $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ for particular value of x from the given data (x_i, y_i), $i = 1, 2 \ldots n$ where $y = f(x)$ is not known explicitly.

The interpolation formula to be used depends on the particular value of x at which the derivatives are required. If the values of x are not equally spaced, we represent the function by newton's divided difference formula and the derivative are obtained.
If the values of x are equally spaced, the derivatives are calculated by using newton's forward (or) Backward interpolation formula.
If the derivatives are required at a point near the beginning of the table, we use newton's forward difference formula, and if the derivatives are required at a point near the end of the table, use Newton's Backward difference formula.

Derivatives using divided differences :-

Find $f'(5)$ and $f''(5)$ using the following data.

x :	0	2	3	4	7	9
$f(x)$:	4	26	58	112	466	922

Soln

Since the values of x are not equally spaced, we shall use Newton's divided difference formula

Divided difference table

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$
0	4			
		11		
2	26		7	
		32		1
3	58		11	
		54		1
4	112		16	
		118		1
7	466		22	
		228		
9	922			

From the above table

$f(x_0) = 4$, $f(x_0, x_1) = 11$, $f(x_0, x_1, x_2) = 7$

$f(x_0, x_1, x_2, x_3) = 1$

Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + f(x_0, x_1, x_2)(x - x_0)(x - x_1) + f(x_0, x_1, x_2, x_3)(x - x_0)(x - x_1)(x - x_2) + \cdots$$

$f(x) = 4 + x(11) + x(x-2)(7) + x(x-2)(x-3)(1)$

$= 4 + 11x + 7x^2 - 14x + x^3 - 5x^2 + 6x$

$\therefore f(x) = x^3 + 2x^2 + 3x + 4 \quad \text{—} (1)$

Diff w.r.t 'x', we get

$f'(x) = 3x^2 + 4x + 3$

$f''(x) = 6x + 4$

put $x = 5$, $f'(5) = 98$

$f''(5) = 34.$

Derivatives using finite differences

Newton's forward difference formula to compute the derivatives:-

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{6p^2-18p+11}{12}\Delta^4 y_0 + \cdots\right]$$

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right]$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right]$$

$$\left.\frac{d^3y}{dx^3}\right|_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$$

Find the first and second derivatives of y at x=15 from the table given below.

x	15	17	19	21	23	25
y	3.873	4.123	4.359	4.583	4.796	5

Soln Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
15	3.873					
		0.250				
17	4.123		-0.014			
		0.236		0.002		
19	4.359		-0.012		-0.001	
		0.224		0.001		0.002
21	4.583		-0.011		0.001	
		0.213		0.002		
23	4.796		-0.009			
		0.204				
25	5					

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 \cdots\right]$$

Here $h = 2$

$$\left.\frac{dy}{dx}\right|_{x=15} = \frac{1}{2}\left[0.250 + \frac{0.014}{2} + \frac{0.002}{3} + \frac{0.001}{4} + \frac{0.002}{5}\right] = 0.1291$$

$$\frac{d^2y}{dx^2} = \frac{1}{4}\left[-0.014 - 0.002 + \frac{11}{12}(-0.001) - \frac{5}{6}(0.002)\right]$$

$$= -\frac{1}{4}\left[0.014 + 0.002 + 0.0009 + 0.0017\right]$$

$$= -0.00460$$

Find the first two derivatives of y at x = 54 from the following table

x	50	51	52	53	54
y	3.6840	3.7084	3.7325	3.7563	3.7798.

Soln Difference table.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
50	3.6840				
		0.0244			
51	3.7084		-0.0003		
		0.0241		0	
52	3.7325		-0.0003		0
		0.0238		0	
53	3.7563		-0.0003		
		0.0235			
54	3.7398				

Newton's Backward difference formula

$$\left.\frac{dy}{dx}\right|_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \cdots\right]$$

$$= \frac{1}{1}\left[0.0235 - \frac{0.0003}{2}\right] = 0.0.02335$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \cdots\right]$$

$$= -0.0003.$$

Find the first and second derivatives of the function tabulated below at $x=0.6$

x :	0.4	0.5	0.6	0.7	0.8
y :	1.5826	1.7974	2.0446	2.3275	2.6511

Solution:-

Since $x=0.6$ is in the middle of the data, we will use Stirling's formula

Difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.4	1.5826				
		0.2138			
0.5	1.7974		0.0330		
		0.2468		0.0035	
0.6	2.0446		0.0365		0.0003.
		0.2833		0.0038	
0.7	2.3275		0.0403		
		0.3236			
0.8	2.6517				

By Stirling's formula

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-3}) + \cdots\right]$$

$$= \frac{1}{0.1}\left[\frac{1}{2}(0.2833 + 0.2468) - \frac{1}{12}(0.0038 + 0.0035)\right]$$

$$= 10\left[0.26505 - 0.00061\right] = 2.6444$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \cdots\right]$$

$$= \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \cdots\right]$$

$$= \frac{1}{0.01}\left[0.0365 - \frac{1}{12}(0.0003)\right]$$

$$\doteq 3.6475.$$

Numerical Integration

(Trapezoidal & Simpson's Rules, Romberg integration).

Introduction

The process of computing the value of a definite integral $\int_a^b y\,dx$ from a set of values (x_i, y_i), $i = 0,1,2,\ldots n$, where $x_0 = a$, $x_n = b$ of the function $y = f(x)$ is called Numerical integration.

Here the function y is replaced by an interpolation formula involving finite difference and then integrated b/n the limits a and b, the value $\int_a^b y\,dx$ is found.

Trapezoidal Rule:-

$$\int_a^b y\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right] \quad \text{--- (2)}$$

$$= \frac{h}{2}\left[\text{Sum of the extreme ordinates} + 2(\text{Sum of the remaining ordinates})\right]$$

Expression (2) is called the trapezoidal rule of the integration.

Truncation error in Trapezoidal Rule:-

The error in Trapezoidal rule is of order h^2 and the total error E is given by

$E = -\frac{(b-a)}{12} h^2 y''(\bar{x})$ where $y''(\bar{x})$ is the largest of $y_0'', y_1'', \ldots y_{n-1}''$.

Simpson's 1/3 rule.

In this rule, the curve through three consecutive points on the graph $y = f(x)$ is replaced by a parabola, a second degree polynomials.

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+y_5+\cdots+y_{n-1}) + 2(y_2+y_4+y_6+\cdots+y_{n-2})\right]$$

This rule is called Simpson's 1/3 rule (or) simply Simpson's rule. It should be noted that for applying this rule, the interval must be divided into even number of subintervals of width h.

Evaluate $\int_0^{\pi} \sin x\,dx$ by dividing the interval into 8 strips using:
(i) Trapezoidal rule and (ii) Simpson's 1/3 rule.

Soln :-

For 8 strips the value of $y = \sin x$ are tabulated as follows.

x	0	$\pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$	$5\pi/8$	$3\pi/4$
$\sin x$	0	0.3827	0.7071	0.9219	1	0.9229	0.7071

x	$7\pi/8$	π
$\sin x$	0.3827	0.

(i) Trapezoidal Rule

$$\int_0^{\pi} \sin x\, dx = \frac{h}{2}\left[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_5 + y_6 + y_7)\right]$$

$$= \frac{\pi}{16}\left[4(0.3827) + 0.7071 + 0.9239 + 2\right]$$

$$= \frac{\pi}{16}\left[10.0548\right]$$

$$= 1.97425$$

(ii) Simpson's 1/3 rule

$$\int_0^{\pi} \sin x\, dx = \frac{h}{3}\left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$= \frac{\pi}{24}\left[4(0.3827) + 0.9239 + 0.9239 + 0.3827) + 2(0.7071 + 1 + 0.7071)\right]$$

$$= \frac{\pi}{24}\left[10.4528 + 2(2.4142)\right]$$

$$= 2.0003.$$

Use Romberg's method to compute $\int_0^1 \frac{dx}{1+x^2}$ correct to 4 decimal places. Hence find an approximate value of π.

Using Trapezoidal rule, the given integral,

$I = \int_0^1 \frac{dx}{1+x^2}$ is evaluated as the

$h = 0.5$, 0.25, and 0.125.

(i) when $h = 0.5$, the value of $y = \frac{1}{1+x^2}$ are tabulated as follows.

x	1	0.5	1.0
y	1	0.8	0.5

Then $I = \frac{0.5}{2}\left[1.5 + 1.6\right] = 0.775$

(ii) when $h = 0.25$ the values of x and y are given as follows.

x :	0	0.25	0.5	0.75	1
y :	1	0.9412	0.8	0.64	0.5

Then $I = \frac{0.25}{2}\left[1.5 + 2(0.9412 + 0.8 + 0.64)\right]$

$= 0.7828.$

(iii) when $h = 0.125$ the values of x and y are obtained as follows,

Numerical Methods

Unit - 2

Interpolation and Approximation

Lagrange's interpolation formula (unequal intervals)

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

---

① Using Lagrange's formula, find the Polynomial to the given data

x	0	1	3
y	5	6	50

Soln

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \cdot y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \cdot y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \cdot y_2$$

Here $x_0 = 0$ $x_1 = 1$ $x_2 = 3$

$y_0 = 5$ $y_1 = 6$ $y_2 = 50$

$$y = f(x) = \frac{(x-1)(x-3)}{(0-1)(0-3)}(5) + \frac{(x-0)(x-3)}{(1-0)(1-3)} \cdot (6)$$

$$+ \frac{(x-0)(x-1)}{} \cdot (50)$$

$$= \frac{(x-1)(x-3)}{3}(5) + \frac{x(x-3)}{-2}(6) + \frac{x(x-1)}{6}(50)$$

$$= \frac{5}{3}[x^2-4x+3] - 3[x^2-3x] + \frac{50}{6}[x^2-x]$$

$$= x^2\left[\frac{5}{3} - 3 + \frac{50}{6}\right] + x\left[-\frac{20}{3} + 9 - \frac{50}{6}\right] + \left[\frac{15}{3}\right]$$

$$= 7x^2 + (-6)x + 5$$

$$\boxed{y = f(x) = 7x^2 - 6x + 5}$$

② Using Lagranges interpolation find $y(2)$ from the following data

x	0	1	3	4	5
y	0	1	81	256	625

Soln

$x_0 = 0 \quad x_1 = 1 \quad x_2 = 3 \quad x_3 = 4. \quad x_4 = 5$

$y_0 = 0 \quad y_1 = 1 \quad y_2 = 81 \quad y_3 = 256 \quad y_4 = 625$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \cdot y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \cdot y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} \cdot y_3$$

Put $x=2$

$$y(2) = \frac{(2-1)(2-3)(2-4)(2-5)}{(0-1)(0-3)(0-4)(0-5)}(0) + \frac{(2-0)(2-3)(2-4)(2-5)}{(1-0)(1-3)(1-4)(1-5)}(1) + \frac{(2-0)(2-1)(2-4)(2-5)}{(3-0)(3-1)(3-4)(3-5)}(81) + \frac{(2-0)(2-1)(2-3)(2-5)}{(4-0)(4-1)(4-3)(4-5)}(256) + \frac{(2-0)(2-1)(2-3)(2-4)}{(5-0)(5-1)(5-3)(5-4)}(625)$$

$$= \frac{(2)(-1)(-2)(-3)}{(1)(-2)(-3)(-4)} + \frac{(2)(1)(-2)(-3)}{(3)(2)(-1)(-2)}(81) + \frac{(2)(1)(-1)(-3)}{(4)(3)(1)(-1)}(256) + \frac{(2)(1)(-1)(-2)}{(5)(4)(2)(1)}(625)$$

$$= \frac{12}{24} + \frac{12}{12}(81) - \frac{6}{12}(256) + \frac{4}{40}(625)$$

$$= \frac{1}{2} + 81 - 128 + 62.5$$

$$= 0.5 + 81 - 128 + 62.5 = 16.$$

3) Use Lagranges Method to find $\log_{10} 656$, given that $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$ and $\log_{10} 661 = 2.8202$.

soln

x	654	658	659	661
$y = \log_{10} x$	2.8156	2.8182	2.8189	2.8202

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0$$
$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

Put $x = 656$

$$y = f(656) = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} \cdot (2.8156)$$
$$+ \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} (2.8182)$$
$$+ \frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)} (2.8189)$$
$$+ \frac{(656-654)(656-659)(656-659)}{(661-654)(661-659)(661-659)} \cdot (2.8202)$$

$$= \frac{(-2)(-3)(-5)}{(-4)(-5)(-7)} (2.8156) + \frac{2(-3)(-5)}{4(-1)(-3)} (2.8182)$$
$$+ \frac{(2)(-2)(-5)}{(5)(1)(-2)} (2.8189) + \frac{(2)(-2)(-3)}{(7)(3)(2)} (2.8202)$$
$$= 0.6033 + 7.0455 - 5.6378 + 0.8058$$
$$= 2.8168.$$

4) Use Lagrange's formula to find the value of y at $x = 6$ from the following data

x :	3	7	9	10

Soln

$x_0 = 3 \quad x_1 = 7 \quad x_2 = 9 \quad x_3 = 10$

$y_0 = 168 \quad y_1 = 120 \quad y_2 = 72 \quad y_3 = 63.$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

Put $x = 6$

$$y = f(6) = \frac{(6-7)(6-9)(6-10)}{(3-7)(3-9)(3-10)}(168) + \frac{(6-7)(6-9)(6-10)}{(7-3)(7-9)(7-10)}(120) + \frac{(6-3)(6-7)(6-10)}{(9-3)(9-7)(9-10)}(72) + \frac{(6-3)(6-7)(6-9)}{(10-3)(10-7)(10-9)}(63)$$

$$= \frac{(-1)(-3)(-4)}{(-4)(-6)(-7)}(168) + \frac{(3)(-3)(-4)}{4(-2)(-3)}(120) + \frac{(3)(-1)(-4)}{(6)(2)(-1)}(72) + \frac{(3)(-1)(-3)}{(7)(3)(1)}(63)$$

$$= 12 + 180 - 72 + 27$$

$$= 147$$

5) Given the values

x	14	17	31	35
$f(x)$	68.7	64.0	44.0	39.1

find $f(27)$ by using Lagrange's interpolation formula.

Soln

$x_0 = 14 \quad x_1 = 17 \quad x_2 = 31 \quad x_3 = 35$

$y_0 = 68.7 \quad y_1 = 64 \quad y_2 = 44 \quad y_3 = 39.1$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

Put $x = 27$

$$y = f(27) = \frac{(27-17)(27-31)(27-35)}{(14-17)(14-31)(14-35)} \cdot (68.7)$$

$$+ \frac{(27-14)(27-31)(27-35)}{(17-14)(17-31)(17-35)} (64.0)$$

$$+ \frac{(27-14)(27-17)(27-35)}{(31-14)(31-17)(31-35)} (44.0)$$

$$+ \frac{(27-14)(27-17)(27-31)}{(35-14)(35-17)(35-31)} (39.1)$$

$$= \frac{(10)(-4)(-8)}{(-3)(-17)(-21)} (68.7) + \frac{13(-4)(-8)}{(3)(-14)(-8)} (64)$$

$$+ \frac{(13)(10)(-8)}{(17)(14)(-4)} (44) + \frac{(13)(10)(-4)}{(21)(18)(4)} (39.1)$$

$$= -20.52 + 35.22 + 48.07 - 13.45$$

6) Find the Missing term in the following table using Lagranges interpolation

x	0	1	2	3	4
y	1	3	9	—	81

Soln

$x_0 = 0 \quad x_1 = 1 \quad x_2 = 3 \quad x_3 = 4$

$y_0 = 1 \quad y_1 = 3 \quad y_2 = 9 \quad y_3 = 81$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3.$$

Put $x = 3$

$$y = f(3) = \frac{(3-1)(3-2)(3-4)}{(0-1)(0-2)(0-4)}(1) + \frac{(3-0)(3-2)(3-4)}{(1-0)(1-2)(1-4)}(3) + \frac{(3-0)(3-1)(3-4)}{(2-0)(2-1)(2-4)}(9) + \frac{(3-0)(3-1)(3-2)}{(4-0)(4-1)(4-2)}(81)$$

$$= \frac{-2}{-8} \cancel{(-3)} + \frac{27}{2} + \frac{81}{4}$$

$$= 31.$$

7) Using Lagranges formula prove

$y_1 = y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} + y_{-5})$

Soln y_{-5}, y_{-3}, y_3, y_5 occur in the answers.

So we can have the table

x	-5	-3	3	5
y	y_{-5}	y_{-3}	y_3	y_5

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_{-5}$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_{-3}$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_3$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_5$$

put $x = 1$

$$y_1 = f(1) = \frac{(1+3)(1-3)(1-5)}{(-5+3)(-5-3)(-5-5)} \cdot y_{-5}$$

$$+ \frac{(1+5)(1-3)(1-5)}{(-3+5)(-3-3)(-3-5)} \cdot y_{-3}$$

$$+ \frac{(1+5)(1+3)(1-5)}{(3+5)(3+3)(3-5)} \, y_3$$

$$+ \frac{(3+5)(3+3)(3-3)}{(5+5)(5+3)(5-3)} \cdot y_5$$

$$= \frac{(4)(-2)(-4)}{(-2)(-8)(-10)} \cdot y_{-5} + \frac{(6)(-2)(-4)}{(2)(-6)(-8)} \cdot y_{-3}$$

$$+ \frac{(6)(4)(-4)}{(8)(6)(-2)} \cdot y_3 + \frac{(6)(4)(-2)}{(10)(8)(2)} \, y_5$$

$= -0.2y \quad + 0.5y \quad + y \quad - 0.3y$

$$+\frac{(0+30)(0+13)(0-18)}{(3+30)(3+13)(3-18)}.(38)$$

$$+\frac{(0+30)(0+13)(0-3)}{(18+30)(18+13)(18-3)}(42)$$

$$=37.23.$$

② Find the value of θ given $f(\theta)=0.3887$ where $f(\theta)=\int_0^\theta \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$ using the table

θ	21°	23°	25°
$f(\theta)$	0.3706	0.4068	0.4433.

Soln

Let $\theta = x$

$f(\theta) = f(x) = y$

x	21°	23°	25°
y	0.3706	0.4068	0.4433

$$x=f(y) = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)}.x_0+\frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)}.x_1$$
$$+\frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)}.x_2$$

Put $y=0.3887$

$$x=f(0.3887)=\frac{(0.3887-0.4068)(0.3887-0.4433)}{(0.3706-0.4068)(0.3706-0.4433)}(21^\circ)$$
$$+\frac{(0.3887-0.3706)(0.3887-0.4433)}{(0.4068-0.3706)(0.4068-0.4433)}(23^\circ)$$
$$+(0.3887-0.3706)(0.3887-0.4068)$$

Newton's divided difference formula : (unequal)

$$y = f(x) = y_0 + (x-x_0)\,\Delta f(x_0) + (x-x_0)(x-x_1)\,\Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-x_2)\,\Delta^3 f(x_0) + \cdots$$

① Using Newton's divided difference formula find $f(x)$ and $f(6)$ from the following data.

x :	1 x_0	2 x_1	7 x_2	8 x_3
$f(x)$:	1	5	5	4

Soln

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1			
		$\frac{5-1}{2-1} = 4$		
2	5		$\frac{0-4}{7-1} = -\frac{4}{6}$	
		$\frac{5-5}{7-2} = 0$		$\frac{-\frac{1}{6}+\frac{4}{6}}{8-1} = \frac{1/2}{7} = \frac{1}{14}$
7	5		$\frac{-1-0}{8-2} = \frac{-1}{6}$	
		$\frac{4-5}{8-7} = -1$		
8	4			

$$y = f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0) + \cdots$$

$$= 1 + (x-1)(4) + (x-1)(x-2)\left(-\frac{4}{6}\right)$$

$$= x^3\left[\frac{1}{14}\right] + x^2\left[\frac{-4}{6}\right] - \frac{3}{14} - \frac{7}{14}\Big]$$

$$+ x\left[4 + \frac{12}{6} + \frac{2}{14} + \frac{21}{14}\right] + \left[1 - 4 - \frac{8}{6} - \frac{14}{14}\right]$$

$$f(x) = \frac{1x^3}{14} - \frac{29}{21}x^2 + \frac{107}{14}x - \frac{16}{63}$$

Put $x = 6$

$$y = f(6) = \frac{1}{14}(6)^3 - \frac{29}{21}(6)^2 + \frac{107}{14}(6) - \frac{16}{36}$$

$$\cancel{= 54 - 114 + 40.4 - 7.833}$$

$$= 15.428 - 49.714 + 45.857 - 0.444$$

$$= 11.127.$$

2) Find $f(x)$ as a polynomial in x for the following data by Newton's divided difference

x	-4	-1	0	2	5
$f(x)$	1245	33	5	9	1335

Soln

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-4	1245				
		$\frac{33-1245}{-1+4} = -404$			
-1	33		$\frac{-28+404}{0+4} = 94$		
		$\frac{5-33}{0+1} = -28$		$\frac{10-94}{2+4} = -14$	
0	5		$\frac{2+28}{2+1} = 10$		$\frac{13+14}{5+4} = 3$
		$\frac{9-5}{2-0} = 2$		$\frac{88-10}{5+1} = 13$	
2	9		$\frac{442-2}{5-0} = 88$		
		$\frac{1335-9}{5-2} = 442$			
5	1335				

$$y = f(x) = f(x_0) + (x-x_0)\,\Delta f(x) + (x-x_0)(x-x_1)\,\Delta^2 f(x)$$
$$+ (x-x_0)(x-x_1)(x-x_2)\,\Delta^3 f(x)$$
$$+ (x-x_0)(x-x_1)(x-x_2)(x-x_3)\,\Delta^4 f(x)$$

$$= 1245 + (x+4)(-404) + (x+4)(x+1)(94) + (x+4)(x+1)(x-0)(x-2)(3)$$
$$+ (x+4)(x+1)(x-0)(-14)$$

$$= 1245 - 404x - 1616 + (x^2+5x+4)\,94 + (x^2+5x+4)(3x^2-6x)$$
$$+ (x^2+5x+4)(-14x)$$

$$= 1245 - 404x - 1616 + 94x^2 + 470x + 376 + 3x^4 + 15x^3 + 12x^2$$
$$-14x^3 - 70x^2 - 56x$$
$$-6x^3 - 30x^2 - 24x$$

$$\cancel{= 1245\ x(-404+470-16-24)} + x^2\cancel{[94-20+12-30]}$$

$$= x^4[3] + x^3[-14+15-6] + x^2[94-70+12-30] + [1245-1616+376]$$
$$+ x[-404+470-56-24]$$

$$= 3x^4 + 5x^3 + 6x^2 - 14x + 5$$

(3) Find the cubic polynomial from the following table using Newton's divided difference formula and hence find $f(4)$

x	$0\ x_0$	$1\ x_1$	$2\ x_2$	$5\ x_3$
y	2	3	12	147

Soln

x	$y=f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	2			
		$\frac{3-2}{1-0}=1$		
1	3		$\frac{9-1}{2-0}=4$	
		$\frac{12-3}{2-1}=9$		$\frac{9-4}{5-0}=1$
2	12		$\frac{45-9}{5-1}=9$	
		$\frac{147-12}{5-2}=45$		
5	147			

$$y=f(x)=y_0+(x-x_0)\Delta f(x)+(x-x_0)(x-x_1)\Delta^2 f(x)+(x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x)$$

$$=2+(x-0)(1)+(x-0)(x-1)(4)+(x-0)(x-1)(x-2)(1)$$

$$=2+x+4x^2-4x+(x^2-x)(x-2)$$

$$=2+x+4x^2-4x+x^3-x^2-2x^2+2x$$

$$=x^3+x^2-x+2$$

Put $x=4$

$$y=f(4)=4^3+4^2-4+2$$

$$=78.$$

Cubic Spline Interpolation formula.

$$g(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} - (x_{i-1} - x)^3 M_i\right] + \frac{1}{h}(x_i - x)\left[y_{i-1} - \frac{h^2}{6}M_{i-1}\right] - \frac{1}{h}(x_{i-1} - x)\left[y_i - \frac{h^2}{6}M_i\right]$$

where, $M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right]$

with $M_0 = M_n = 0$

(1) Obtain cubic spline polynomial which best fits with the following data, given that $y_0'' = y_3'' = 0$

x	-1 x_0	0 x_1	1 x_2	x_3 2
y	-1 y_0	1 y_1	3 y_2	35 y_3

Soln

Given $M_0 = M_3 = 0$, $h = 1$

WKT $M_{i-1} + 4M_i + M_{i+1} = 6\left[y_{i-1} - 2y_i + y_{i+1}\right]$

Put $i = 1$

$$M_0 + 4M_1 + M_2 = 6\left[y_0 - 2y_1 + y_2\right]$$

$$4M_1 + M_2 = 6\left[-1 - 2 + 3\right]$$

$$4M_1 + M_2 = 0 \quad \text{——— (1)}$$

Put $i = 2$

$$M_1 + 4M_2 + M_3 = 6\left[y_1 - 2y_2 + y_3\right]$$

$$M_1 + 4M_2 = 6\left[1 - 6 + 35\right]$$

Solve ① & ②

$M_1 = -12 \quad M_2 = 48$

The cubic spline polynomial is

$$S(x) = \frac{1}{6}\left[(x_i - x)^3 M_{i-1} - (x_{i-1} - x)^3 M_i \right]$$
$$+ (x_i - x)\left[y_{i-1} - \frac{1}{6} M_{i-1} \right]$$
$$- (x_{i-1} - x)\left[y_i - \frac{1}{6} M_i \right]$$

Case(i) $-1 < x < 0$

Put $i = 1$

$$S(x) = \frac{1}{6}\left[(x_1 - x)^3 M_0 - (x_0 - x)^3 M_1 \right]$$
$$+ (x_1 - x)\left[y_0 - \frac{1}{6} M_0 \right] - (x_0 - x)\left[y_1 - \frac{1}{6} M_1 \right]$$

$$= \frac{1}{6}\left[-(-1-x)^3(-12) \right] + (0-x)(-1)$$
$$- (-1-x)\left[1 + \frac{12}{6} \right]$$

$$= \frac{1}{6}\left[-12(1+x)^3 \right] + x + (1+x)(3)$$

$$= -2\left[1 + x^3 + 3x + 3x^2 \right] + x + 3 + 3x$$

$$= -2 - 2x^3 - 6x - 6x^2 + x + 3 + 3x$$

$$\boxed{S(x) = -2x^3 - 6x^2 - 2x + 1, \quad -1 < x < 0}$$

Case(ii) $0 < x < 1$

Put $i = 2$

$$S(x) = \frac{1}{6}\left[(x_2 - x_1)^3 M_1 - (x_1 - x)^3 M_2 + (x_2 - x)\left[y_1 - \frac{1}{6}M_1\right] - (x_1 - x)\left[y_2 - \frac{1}{6}M_2\right]\right]$$

$$= \frac{1}{6}\left[(1-x)^3(-12) - (0-x)^3(48)\right] + (1-x)\left[1 - \frac{1}{6}(-12)\right] - (0-x)\left[3 - \frac{1}{6}\times 48\right]$$

$$= \frac{1}{6}\left[-12(1-x)^3 + 48x^3\right] + 3(1-x) - 5x$$

$$= \frac{1}{6}\left[-12(1 - x^3 - 3x + 3x^2) + 48x^3\right] + 3 - 3x - 5x$$

$$= \frac{1}{6}\left[-12 + 12x^3 + 36x - 36x^2 + 48x^3\right] + 3 - 3x - 5x$$

$$= x^3[2 + 8] + x^2[-6] + x[6 - 8] - 2 + 3$$

$$\boxed{S(x) = 10x^3 - 6x^2 - 2x + 1, \quad 0 < x < 1}$$

Case (iii) $1 < x < 2$

Put $i = 3$

$$S(x) = \frac{1}{6}\left[(x_3 - x)^3 M_2 - (x_2 - x)^3 M_3\right] + (x_3 - x)\left[y_2 - \frac{1}{6}M_2\right] - (x_2 - x)\left[y_3 - \frac{1}{6}M_3\right]$$

$$= \frac{1}{6}\left[(2-x)^3 48\right] + (2-x)\left[3 - \frac{1}{6}\times 48\right]$$

$= 8(2-x)^3 + (2-x)(-5) - 35(1-x)$

$= 8[8 - x^3 - 12x + 6x^2] - 10 + 5x - 35 + 35x$

$= 64 - 8x^3 - 96x + 48x^2 - 10 + 5x - 35 + 35x$

$$\boxed{S(x) = -8x^3 + 48x^2 - 56x + 19, \quad 1 < x < 2}$$

The cubic Spline Polynomial is

$$S(x) = \begin{cases} -2x^3 - 6x^2 - 2x + 1, & -1 < x < 0 \\ 10x^3 - 6x^2 - 2x + 1, & 0 < x < 1 \\ -8x^3 + 48x^2 - 56x + 19, & 1 < x < 2 \end{cases}$$

(2) From the following table

x	1 x_0	2 x_1	3 x_2
y	-8 y_0	-1 y_1	18 y_2

Compute $y(1.5)$ and $y'(1)$ using cubic Spline.

Soln

Take $M_0 = M_2 = 0$, $h = 1$

W.K.T $M_{i-1} + 4M_i + M_{i+1} = 6[y_{i-1} - 2y_i + y_{i+1}]$

Put $i = 1$

$M_0 + 4M_1 + M_2 = 6[y_0 - 2y_1 + y_2]$

$4M_1 = 6[-8 + 2 + 18]$

$4M_1 = 72$

$\boxed{M_1 = 18}$

The cubic spline polynomial is

$$S(x) = \frac{1}{6}\left[(x_i - x)^3 M_{i-1} - (x_{i-1} - x)^3 M_i\right] + (x_i - x)\left[y_{i-1} - \frac{1}{6}M_{i-1}\right] - (x_{i-1} - x)\left[y_i - \frac{1}{6}M_i\right]$$

Case (i) $1 < x < 2$

Put $i = 1$

$$S(x) = \frac{1}{6}\left[(x_1 - x)^3 M_0 - (x_0 - x)^3 M_1\right] + (x_1 - x)\left[y_0 - \frac{1}{6}M_0\right] - (x_0 - x)\left[y_1 - \frac{1}{6}M_1\right]$$

$$= \frac{1}{6}\left[(2-x)^3 (0) - (1-x)^3 (18)\right] + (2-x)\left[-8 - \frac{1}{6}(0)\right] - (1-x)\left[-1 - \frac{\cancel{x}(18)}{6}\right]$$

$$= \frac{1}{6}\left[-(1-x)^3(18)\right] + (2-x)(-8) - (1-x)\left[-1-3\right]$$

$$= -18(1-x)^3 - 8(2-x) + 4(1-x)$$

$$= -18(1-x)^3 - 16 + 8x + 4 - 4x$$

$$\boxed{S(x) = -18(1-x)^3 + 4x - 12, \quad 1 < x < 2}$$

Put $x = 1.5$

$$y(1.5) = S(1.5) = -18(1-1.5)^3 + 4(1.5) - 12$$

$$= -5.625$$

$y'(1) = 9(0) + 4 = 4$

$\boxed{y'(1) = 4}$ $\boxed{y(1.5) = -5.625}$

(3) Find the cubic Spline interpolation

x :	1	2	3	4	5
f :	1 y_0	0 y_1	1 y_2	0 y_3	1 y_4

soln

Take $M_0 = M_4 = 0$, $h = 1$

WKT
$$M_{i-1} + 4M_i + M_{i+1} = 6[y_{i-1} - 2y_i + y_{i+1}]$$

Put $i=1$
$$M_0 + 4M_1 + M_2 = 6[y_0 - 2y_1 + y_2]$$
$$4M_1 + M_2 = 6[1-0+1] = 12$$
$$4M_1 + M_2 = 12 \quad \text{——— (1)}$$

Put $i = 2$
$$M_1 + 4M_2 + M_3 = 6[y_1 - 2y_2 + y_3]$$
$$= 6[0-2+0]$$
$$M_1 + 4M_2 + M_3 = -12 \quad \text{——— (2)}$$

Put $i = 3$
$$M_2 + 4M_3 + M_4 = 6[y_2 - 2y_3 + y_4]$$
$$M_2 + 4M_3 = 6[1-0+1]$$
$$M_2 + 4M_3 = 12 \quad \text{——— (3)}$$

from (1) & (2)

$4 \times (1) \Rightarrow 16M_1 + \cancel{4M_2} = 48$

from (2) & (3)

(2) $\Rightarrow M_1 + \cancel{4M_2} + M_3 = -12$

$4 \times (3) \Rightarrow \cancel{4M_2} + 16M_3 = 48$

$M_1 - 15M_3 = -60$ —— (5)

Solve (4) & (5)

$$\boxed{M_3 = \frac{30}{7}}$$

(5) $\Rightarrow M_1 = -60 + 15M_3$

$M_1 = -60 + \frac{450}{7}$

$$\boxed{M_1 = \frac{30}{7}}$$

(1) $\Rightarrow 4M_1 + M_2 = 12$

$M_2 = 12 - 4M_1$

$= 12 - 4\left(\frac{30}{7}\right)$

$$\boxed{M_2 = \frac{-36}{7}}$$

The cubic spline polynomial is

$$S(x) = \frac{1}{6}\left[(x_i - x)^3 M_{i-1} - (x_{i-1} - x)^3 M_i\right] + (x_i - x)\left[y_{i-1} - \frac{1}{6}M_{i-1}\right] - (x_{i-1} - x)\left[y_i - \frac{1}{6}M_i\right]$$

Case (i) $-1 < x < 2$

Put $i = 1$

$$S(x) = \frac{1}{6}\left[(x_1 - x)^3 M_0 - (x_0 - x)^3 M_1\right] + (x_1 - x)\left[y_0 - \frac{1}{6}M_0\right] - (x_0 - x)\left[y_1 - \frac{1}{6}M_1\right]$$

$$= \frac{1}{6}\left[(2-x)^3(0) - (1-x)^3\left(\frac{30}{7}\right)\right] + (2-x)\left[1 - \frac{1}{6}(0)\right]$$

$(a-b)^3$
$= a^3 - 3$

$$= \frac{1}{6}\left[-(1-x)^3\left(\frac{30}{7}\right)\right] + (2-x)[1]$$
$$- (1-x)\left[-\frac{1}{6}\,\frac{30}{7}\right]$$

$$= \frac{1}{6}\left[-\frac{30}{7}(1-x)^3\right] + (2-x) + \frac{5}{7}(1-x)$$

$$= \frac{1}{6}\left[-\frac{30}{7}\left[1^3 - x^3 - 3x + 3x^2\right]\right] + 2 - x + \frac{5}{7} - \frac{5}{7}x$$

$$= \cancel{-\frac{5}{7}} + \frac{5}{7}x^3 + \frac{15}{7}x + 15x^2 + 2 - x + \cancel{\frac{5}{7}} - \frac{5}{7}x$$

$$= \frac{5}{7}x^3 + 15x^2 + x\left(\frac{15}{7} - 1 - \frac{5}{7}\right)$$

$$S(x) = \frac{5}{7}x^3 + 15x^2 + \frac{3}{7}x + 2, \quad 1 \le x \le 2$$

Case (ii) $2 < x < 3$.

Put $i = 2$.

$$S(x) = \frac{1}{6}\left[(x_2 - x)^3 M_1 - (x_1 - x)^3 M_2\right]$$
$$+ (x_2 - x)\left[y_1 - \frac{1}{6}M_1\right]$$
$$- (x_1 - x)\left[y_2 - \frac{1}{6}M_2\right]$$

$$= \frac{1}{6}\left[(3-x)^3\frac{30}{7} - (2-x)\left(\frac{-36}{7}\right)\right]$$
$$+ (3-x)\left[0 - \frac{1}{6}\left(\frac{30}{7}\right)\right]$$
$$- (2-x)\left[1 - \frac{1}{6}\left(-\frac{36}{7}\right)\right]$$

$$= \frac{1}{6}\left[\frac{30}{7}(3-x)^3 + \frac{36}{7}(2-x) \right] + (3-x)\left(-\frac{5}{7}\right)$$

$$-(2-x)\left[1 + \frac{5}{7} \right]$$

$$= \frac{5}{7}\left[27 - 27x + 9x^2 - x^3 \right] + \frac{6}{7}\left[4 + x^2 - 4x \right]$$

$$-\frac{15}{7} + \frac{5x}{7} - \frac{26}{7} + \frac{13x}{7}$$

$$= x^3\left[-\frac{5}{7}\right] + x^2\left[\frac{45}{7} + \frac{6}{7} + \cancel{\frac{5}{7}} \cancel{+\frac{13}{7}} \right]$$

$$+ x\left[-135 - \frac{24}{7} + \frac{5}{7} + \frac{13}{7}\right] + \frac{135}{7} + \frac{24}{7} - \frac{15}{7}$$

$$-\frac{26}{7}$$

$$S(x) = -\frac{5}{7}x^3 + \frac{51}{7}x^2 - \frac{951}{7} + \frac{118}{7}, \quad 2 < x < 3$$

case (iii) $3 < x < 4$

Put $i = 3$.

$$S(x) = \frac{1}{6}\left[(x_3 - x)^3 M_2 - (x_2 - x) M_3 \right]$$

$$+ (x_3 - x)\left[y_2 - \frac{1}{6}M_2 \right] - (x_2 - x)\left[y_3 - \frac{1}{6}M_3 \right]$$

$$= \frac{1}{6}\left[(4-x)^3\left(-\frac{36}{7}\right) + (3-x)^3\left(\frac{30}{7}\right) \right]$$

$$+ (4-x)\left[1 - \frac{1}{6}\left(-\frac{36}{7}\right) \right]$$

$$-(3-x)\left[0 - \frac{1}{6}\left(\frac{30}{7}\right) \right]$$

$$= \frac{1}{6}\Big[-\frac{36}{7}[64 - 48x + 12x^2 - x^3]$$

$$-\frac{30}{7}[27 - 27x + 9x^2 - x^3]\Big]$$

$$+(4-x)\left(1+\frac{6}{7}\right) - (3-x)\left(-\frac{5}{7}\right)$$

$$= \cancel{\Big[-\frac{384}{7}} \quad {}^{13}/_{7}$$

$$= \frac{1}{7}\Big[-384 + 288x - 72x^2 + 6x^3 - 810$$

$$+ 810x - 270x^2 + 30x^3$$

$$+52 - 13x + 15 - 5x\Big]$$

$$= \frac{1}{7}\Big[x^3[30+6] + x^2[-72-270] +$$

$$+ x[288 + 810 - 13 - 5] +$$

$$[-384 - 810 + 52 + 15]\Big]$$

$$S(x) = \frac{1}{7}\Big[36x^3 - 342x^2 + 1080x - 1127, \quad 3 \le x \le 4$$

case (iv) $4 < x < 5$

put $i = 4$.

$$S(x) = \frac{1}{6}\Big[(x_3 - x)^3 M_3 - (x_2 - x) M_4\Big]$$

$$+ (x_4 - x)\Big[y_3 - \frac{1}{6}M_3\Big]$$

$$-(x_3 - x)\Big[y_4 - \frac{1}{6}M_4\Big]$$

$$= \frac{1}{6}\Big[(5-x)^3\left(\frac{30}{7}\right) - 0\Big] + (5-x)\Big[0 - \frac{1}{6}\left(\frac{30}{7}\right)\Big]$$

$$+(x-4)[1-0]$$

4) Find the cubic spline for the data

x	1	2	3
y	-6	-1	16

Hence evaluate $y(1.5)$ given that $y_0'' = y_2'' = 0$.

Soln

Given $h = 1$ $\quad M_0 = M_2 = 0$

W.K.T

$$M_{i-1} + 4M_i + M_{i+1} = 6\left[y_{i-1} - 2y_i + y_{i+1}\right]$$

Put $i = 1$

$$M_0 + 4M_1 + M_2 = 6\left[y_0 - 2y_1 + y_2\right]$$

$$4M_1 = 6\left[-6 - 2(-1) + 16\right]$$

$$4M_1 = 72$$

$$\boxed{M_1 = 18}$$

The cubic spline polynomial is

$$S(x) = \frac{1}{6}\left[(x_i - x)^3 M_{i-1} - (x_{i-1} - x)^3 M_i\right]$$
$$+ (x_i - x)\left[y_{i-1} - \frac{1}{6}M_{i-1}\right]$$
$$- (x_{i-1} - x)\left[y_i - \frac{1}{6}M_i\right]$$

Case (i) $1 \le x \le 2$

Put $i = 1$

$$S(x) = \frac{1}{6}\left[(x_1 - x)^3 M_0 - (x_0 - x)^3 M_1\right]$$

$$= \frac{1}{6}\left[(2-x)^3(0) + (x-1)^3(18)\right]$$
$$+ (2-x)\left[-6 - \frac{1}{6}(0)\right]$$
$$+ (x-1)\left[-1 - \frac{1}{6}(18)\right]$$
$$= \frac{1}{6}\left[(x-1)^3(18)\right] + (2-x)(-6-0)$$
$$+ (x-1)(-1-3)$$
$$= 3(x^3 - 3x^2 + 3x - 1) - 12 + 6x - 4x + 4$$
$$= 3x^3 - 9x^2 + 11x - 11$$

$S(x) = 3x^3 - 9x^2 + 11x - 11$

Case (ii) $2 \le x \le 3$

Put $i = 2$.

$$S(x) = \frac{1}{6}\left[(x_2 - x)^3 M_1 - (x_1 - x)^3 M_2\right]$$
$$+ (x_2 - x)\left[y_1 - \frac{1}{6}M_1\right]$$
$$- (x_1 - x)\left[y_2 - \frac{1}{6}M_2\right]$$
$$= \frac{1}{6}\left[(3-x)^3\, 18 - (2-x)^3(0)\right]$$
$$+ (3-x)\left[-1 - \frac{1}{6}(18)\right]$$
$$- (x-2)\left[16 - \frac{1}{6}(0)\right]$$
$$= \frac{18}{6}\left[27 - 27x + 9x^2 - x^3\right]$$
$$-12 + 4x + 16x - 32$$

$$S(x) = -3x^3 + 27x^2 - 61x + 37$$

$$y = S(x) = \begin{cases} 3x^3 - 9x^2 + 11x - 11, & 1 \le x \le 2 \\ -3x^3 + 27x^2 - 61x + 37, & 2 \le x \le 3 \end{cases}$$

To find $y(1.5)$

$$S(1.5) = 3(1.5)^3 - 9(1.5)^2 + 11(1.5) - 11$$

$$= -4.625$$

Newton's forward interpolation formula (equal intervals).

$$y = f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

where $u = \dfrac{x - x_0}{h}$

(1) Using Newton's forward interpolation formula, find the polynomial $f(x)$ satisfying the following data. Hence evaluate y at $x = 5$.

x	4	6	8	10
y	1	3	8	10

Soln

$$u = \frac{x - x_0}{h}, \quad h = 2$$

$$u = \frac{x-4}{2}$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
6	3		3	
		5		-6
8	8		-3	
		2		

The Newton's forward interpolation form is

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 1 + \left(\frac{x-4}{2}\right)(2) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2}-1\right)}{2} \times 3 + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2}-1\right)\left(\frac{x-4}{2}-2\right)}{3!} \times -6$$

$$= 1 + (x-4) + \frac{3(x-4)(x-6)}{8} - \frac{(x-4)(x-6)(x-8)}{8}$$

$$= \frac{1}{8}\left[8 + 8x - 32 + 3\left[x^2 - 10x + 24\right] - \left[x^3 - 18x^2 + 104x - 192\right]\right]$$

$$y = \frac{1}{8}\left[-x^3 + 21x^2 - 126x + 240\right]$$

Put $x = 5$

$$y(5) = \frac{1}{8}\left[-5^3 + 21 \times 5^2 - 126 \times 5 + 240\right]$$

$$\boxed{y(5) = 1.25}$$

(2) Fit a polynomial, by using Newtons forward interpolation formula to the data given below.

x	0 x_0	1 x_1	2 x_2	3 x_3
y	1 y_0	2 y_1	1 y_2	10 y_3

Soln

$$u = \frac{x - x_0}{h}, \quad h = 1$$

$$u = \frac{x - 0}{1} = x$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
			-2	
1	2			12
		-1		
2	1		10	
		9		
3	10			

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 1 + \frac{x}{1!}(2) + \frac{x(x-1)}{2!}(-1) + \frac{x(x-1)(x-2)}{3!}(10)$$

$$= 1 + 2x + \frac{(x^2 - x)}{2} + \frac{10}{6}\left[(x^2 - x)(x-2)\right]$$

$$= 1 + 2x + \frac{x^2}{2} - \frac{x}{2} + \frac{5}{3}\left[x^3 - 2x^2 - x^2 + 2x\right]$$

$$= \frac{5}{3}x^3 + x^2\left[\frac{1}{2} - \frac{10}{3}\right] + x\left[2 - \frac{1}{2} + \frac{10}{3}\right] + 1$$

(3) From the data given below find the number of students whose weight is between 60 to 70.

weight in kg's	0-40	40-60	60-80	80-100	100-120
No. of Students	250	120	100	70	50

Soln

$$u = \frac{x - x_0}{h}, \quad h = 20$$

$$u = \frac{x - 40}{20}$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
		120			
Below 60	310		-20		
		100		-10	
Below 80	410		-30		20
		70		10	
Below 100	540		-20		
		50			
Below 120	590				

The Newton's forward interpolation formula is

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+ \frac{u(u-1)(u-2)(u-3)}{} \Delta^4 y_0 + \cdots$$

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \dots$$

$$= 250 + \left(\frac{x-40}{20}\right)120 + \frac{\left(\frac{x-40}{20}\right)\left(\frac{x-40}{20}-1\right)}{2} \times -20$$

$$+ \frac{\left(\frac{x-40}{20}\right)\left(\frac{x-40}{20}-1\right)\left(\frac{x-40}{20}-2\right)}{6} \times -10$$

$$+ \frac{\left(\frac{x-40}{20}\right)\left(\frac{x-40}{20}-1\right)\left(\frac{x-40}{20}-2\right)\left(\frac{x-40}{20}-3\right)}{24} \times 20$$

$$y = 250 + 6(x-40) - 10\left(\frac{x-40}{20}\right)\left(\frac{x-60}{20}\right)$$

$$-\frac{5}{3}\left(\frac{x-40}{20}\right)\left(\frac{x-60}{20}\right)\left(\frac{x-80}{20}\right)$$

$$+\frac{5}{6}\left(\frac{x-40}{20}\right)\left(\frac{x-60}{20}\right)\left(\frac{x-80}{20}\right)\left(\frac{x-100}{20}\right)$$

$$y(70) = 250 + 6(70-40) - 10\left(\frac{70-40}{20}\right)\left(\frac{70-60}{20}\right) - \frac{5}{3}\left(\frac{70-40}{20}\right)\left(\frac{70-60}{20}\right)\left(\frac{70-80}{20}\right)$$

$$+\frac{5}{6}\left(\frac{70-40}{20}\right)\left(\frac{70-60}{20}\right)\left(\frac{70-80}{20}\right)\left(\frac{70-100}{20}\right)$$

$$= 250 + 180 - \frac{15}{2} + \frac{5}{8} + \frac{15}{32}$$

$$y(70) = 423.59 \simeq 424.$$

$$y(60) = 370.$$

No. of Students whose weight between 60–70 $\Big\} = y(70) - y(60)$

$= 424 - 370$

Newton's Backward Interpolation formula

$$y = y_0 + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n$$

Where $v = \frac{x - x_n}{h}$

(1) Use Newton's backward difference formula to construct an interpolating polynomial of degree 3 for the data.

$f(-0.75) = -0.0718125$0, $f(-0.5) = -0.024750$

$f(-0.25) = 0.33493750$, $f(0) = 1.10100$.

Hence find $f(-\frac{1}{3})$.

soln.

$$v = \frac{x - x_n}{h} \quad \& \quad h = 0.25$$

$$v = \frac{x - 0}{0.25} = \frac{x}{0.25}$$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
-0.75	$-0.0718125$0			
		0.0470625		
-0.50	-0.024750		0.312625	
		$0.359687$5		0.09375
-0.25	0.33493750		0.406375	
		0.7660625		
0	1.10100			

The Newton's backward interpolation formula is

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$

$$= 1.10100 + \left(\frac{x}{0.25}\right)(0.7660625)$$
$$+ \left(\frac{x}{0.25}\right)\left(\frac{x}{0.25}+1\right)(0.406375)$$
$$+ \frac{\left(\frac{x}{0.25}\right)\left(\frac{x}{0.25}+1\right)\left(\frac{x}{0.25}+2\right)}{3!}(0.09375)$$

$$= 1.10100 + (-1.33333)(0.7660625)$$
$$+ \frac{(-1.33333)(-0.33333)}{2}(0.40637)$$
$$+ \frac{(-1.33333)(-0.33333)(-0.66666)}{6}(0.09375)$$

$$= 1.10100 - 1.021414 + 0.090304426 + 0.0046295$$

$$y(-1/3) = 0.165260.$$

② The amount A of a substance remaining in a reacting system after an interval of time t in a certain chemical experiment

T (min)	2	5	8	11
A (gm)	94.8	87.9	81.3	75.1

Obtain the value of A where $t = 9$ mins using Newton's interpolation formula.

Soln

T x	A y	∇y	$\nabla^2 y$	$\nabla^3 y$
2	94.8			
		-6.9		
5	87.9		0.3	
		-6.6		0.1
8	81.3		0.4	
		-6.2		
11	75.1			

$$v = \frac{x - x_n}{h}, \quad h = 3$$

The Newton's Backward interpolation formula is

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$

$$y = 75.1 + \left(\frac{x-11}{3}\right)(-6.2) + \frac{\left(\frac{x-11}{3}\right)\left(\frac{x-11}{3}+1\right)}{}(0.4)$$

$$y = 75.1 - 6.2\left(\frac{x-11}{3}\right) + \left(\frac{(x-11)(x-8)}{8}\right) \times 0.4$$

$$+ \frac{(x-11)(x-8)(x-5)}{162} \times 0.1$$

Put $x = 9$

$$y(9) = 75.1 - \frac{6.2(9-11)}{3} + \frac{(9-11)(9-8)}{18} \times 0.5$$

$$+ \frac{(9-11)(9-8)(9-5)}{162} \times 0.1$$

$$= 75.1 + \frac{62}{15} - \frac{2}{45} - \frac{2}{405}$$

$$y(9) = 79.1839.$$

Solution of Equations and Eigen value problems

Introduction:-

The equation of the form $f(x)=0$ is algebraic equation if $f(x)$ is purely polynomial in x. For example $2x^3-6x^2+x-1=0$, $x^2+2x-2=0$ are algebraic equation.

If $f(x)$ also contains trigonometric, logarithmic, exponential functions, then the equation $f(x)=0$ is known as transcendental equation.

Methods for solving the equation $f(x)=0$

1. Method of false position
2. Iteration method
3. Newton Raphson method
4. Method of Bisection.

Iteration method (or) method of successive approximation (or) Fixed point method.

For solving the equation $f(x)=0$, we start with approximate value of the root. The equation $f(x)=0$ is expressed as $x=\phi(x)$. The equation $x=\phi(x)$ is called fixed point equation.

If x_0 is the starting the approximate value of the actual root α, of $x = \phi(x)$, the first approximation is $x_1 = \phi(x_0)$. putting $x = x_1$ in the RHS of $x = \phi(x)$, the second approximation is $x_2 = \phi(x_1)$. proceeding this way, the successive approximations are. $x_1 = \phi(x_0)$, $x_2 = \phi(x_1)$ ... $x_n = \phi(x_{n-1})$. The sequence of approx roots are $x_1, x_2 \cdots x_n$. If its converge to 'α' is taken as the root of the eqn $f(x) = 0$.

$\therefore$ The Iteration formula is given by $x_{n+1} = \phi(x_n)$, $n = 0, 1, 2 \cdots$

Solve the equation $x^3 + x^2 - 1 = 0$ for the positive root by iteration method, correct to four decimal places.

Soln

Let $f(x) = x^3 + x^2 - 1 \quad - (1)$

$f(0) = -1 \quad -ve$

$f(1) = 1 \quad +ve$

$\therefore$ the root lies b/n 0 and 1

$x^3 + x^2 - 1 = 0$

$\Rightarrow x^2(x+1) = 1$

$\Rightarrow x^2 = \frac{1}{x+1}$

$\Rightarrow x = \frac{1}{\sqrt{x+1}}$

The given equation expressed as $x = \phi(x)$ where $\phi(x) = \frac{1}{\sqrt{1+x}}$

$$\phi'(x) = \frac{1}{2}\frac{1}{(1+x)^{3/2}}$$

$$|\phi'(0)| = \frac{1}{2} \text{ and } |\phi'(1)| = \frac{1}{4\sqrt{2}} < 1$$

$$\therefore |\phi'(x)| < 1 \quad \forall x \in (0,1)$$

Choosing $x_0 = 0.75$, the successive approximations are

$$x_1 = \frac{1}{\sqrt{1+0.75}} = 0.75593$$

$$x_2 = 0.75465$$

$$x_3 = 0.75493$$

$$x_4 = 0.75487$$

$$x_5 = 0.75488$$

$$x_6 = 0.75488$$

Hence the root is 0.7549.

Find the real root of the equation $3x - \cos x - 2 = 0$ by iteration method correct to three decimal places.

Soln

$$f(x) = 3x - \cos x - 2$$

$$f(0) = -3 \text{ (-ve)}$$

$f(1) > 0$, The root of the equation $f(x) = 0$ lies b/w 0 and 1

The given equation $f(x) = 0$ may be written as $x = \frac{2+\cos x}{3} = \phi(x)$.

$|\phi'(x)| < 1 \ \forall x \in (0,1)$

we choose initial approximation

$x_0 = 0.75$

The successive approximation are

$x_1 = \frac{1}{3}(2+\cos(0.75)) = 0.9106$

$x_2 = \frac{1}{3}[2+\cos(0.9106)] = 0.8711$

$x_3 = \frac{1}{3}[2+\cos(0.8711)] = 0.8813$

$x_4 = \frac{1}{3}[2+\cos(0.8813)] = 0.8787$

$x_5 = \frac{1}{3}[2+\cos(0.8787)] = 0.8794$

$x_6 = \frac{1}{3}[2+\cos(0.8794)] = 0.8792$

$x_7 = \frac{1}{3}[2+\cos(0.8792)] = 0.8793$

$\therefore$ The root of the eqn $f(x) = 0$ is 0.879, correct to 3-decimal places

Newton's method (or) Newton-Raphson method (or) method of tangents.

This method starts with an initial approximation to the root of an equation, a better and closer approximation to the root can be found by using an iterative process.

Use Newton's - Raphson method to solve the equation $3x - \cos x - 1 = 0$

Soln

Let $f(x) = 3x - \cos x - 1$

$f'(x) = 3 + \sin x$

Therefore, by Newton's - Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow x_{n+1} = x_n - \frac{(3x_n - \cos x_n - 1)}{3 + \sin(x_n)}$$

Since $f(0) = -2 < 0$ and $f(1) = 1.4597 > 0$

$\therefore$ The root of $f(x) = 0$ lies b/n 0 and 1

We choose initial approximation

$$x_0 = 0.6$$

First approximation,

$$x_1 = x_0 + \frac{(3x_0 - \cos x_0 - 1)}{3 + \sin(x_0)}$$

$$= 0.6 + \frac{(3(0.6) - \cos(0.6) - 1)}{3 + \sin(0.6)}$$

$$= 0.6071$$

Second approximation -

$$x_2 = x_1 - \frac{(3x_1 - \cos x_1 - 1)}{3 + \sin(x_1)}$$

$$= 0.6071 - \frac{0.000028}{3.5707} = 0.6071$$

Show that the iteration formula for finding the square root of N is

$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$ and hence find the value of $\sqrt{15}$

Solution:-

Let $f(x) = x^2 - N$

$\therefore f'(x) = 2x$

Newton's formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2 \ldots$$

$$\therefore x_{n+1} = x_n - \frac{(x_n^2 - N)}{2x_n}$$

$$= \frac{2x_n^2 - x_n^2 - N}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + N}{2x_n} = \left(x_n + \frac{N}{x_n}\right)\frac{1}{2}$$

To find $\sqrt{15}$

The initial approximation $x_0 = 3.5$

Taking $N = 15$, the successive approximation are,

$$x_1 = \frac{1}{2}\left[3.5 + \frac{15}{3.5}\right] = 3.893$$

$$x_2 = \frac{1}{2}\left[3.893 + \frac{15}{3.893}\right] = 3.873$$

$$x_3 = \frac{1}{2}\left[3.873 + \frac{15}{3.873}\right] = 3.873$$

$\therefore$ The square root of 15, correct to 3 decimal place is 3.873.

Bisection Method (or) Interval Halving Method.

Let $f(x)=0$ be the given equation.
If $f(a)$ and $f(b)$ are of opposite sign at least one real root of $f(x)=0$ lies between 'a' and 'b'.
We assume that $f(a)>0$ and $f(b)<0$.
The first approximation to the root is computed as $x_0 = \frac{a+b}{2}$
If $f(x_0)<0$, the root lies in the interval $[a, x_0]$. In this case, the second approx is $x_1 = \frac{a+x_0}{2}$. The procedure is continued until the approximate roots coincide with exact root.

The method is very simple and the convergence of this method is slow but sure.

Find the positive root of $x^3-x-1=0$, correct to four decimal's

Solution,

$$\text{Let } f(x) = x^3 - x - 1$$
$$f(1) = -1 < 0$$
$$f(1.5) = 0.8750 > 0$$

Hence the root is lies b/n 1 and 1.5

The first approximation

$$x_0 = \frac{1+1.5}{2} = 1.25$$
$$f(x_0) = f(1.25) = -0.297 < 0$$

The root interval is [1.25, 1.5]

The second approximation

$$x_1 = \frac{1.25 + 1.5}{2} = 1.375$$

$f(1.375) = 0.225 > 0$

$\therefore$ The root interval is [1.25, 1.375]

The approximation

$$x_2 = \frac{1.25 + 1.375}{2} = 1.3125$$

$f(1.3125) = -0.0515 < 0$

$\therefore$ The root interval is [1.3125, 1.375]

$$x_3 = \frac{1.3125 + 1.375}{2} = 1.3438$$

and $f(1.3438) = 0.083 > 0$

The root interval is [1.3125, 1.3438]

$$x_4 = \frac{1.3125 + 1.3438}{2} = 1.3282$$

$\therefore$ The root interval is [1.3125, 1.3282]

$$x_5 = \frac{1.3125 + 1.3282}{2} = 1.3204$$

Now $f(1.3204) = -0.018 < 0$

The root interval is [1.3204, 1.3282]

$$x_6 = \frac{1.3204 + 1.3282}{2}$$

$x_6 = 1.3243$

$x_7 = 1.1.3263$, $x_8 = 1.3253$

$x_9 = 1.3248$, $x_{10} = 1.32455$, $x_{11} = 1.3242$

$x_{12} = 1.32425$

$\therefore$ The approximate root is 1.3242

(1) Solve the following system of equations using Gauss elimination method.

$x+y+z=9,\ 2x-3y+4z=13,\ 3x+4y+5z=40$

Sol:

The augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -1 & \frac{2}{5} & -1 \\ 0 & 1 & 2 & 13 \end{array}\right] \quad R_2 \to \frac{R_2}{5}$$

$$\sim \left[\begin{array}{cccc} 1 & 1 & 1 & 9 \\ 0 & 1 & \frac{2}{5} & -1 \\ 0 & 0 & \frac{12}{5} & 12 \end{array}\right] \quad R_3 \to R_3 + R_2$$

From the last step.

$x+y+z=9$ — (1)

$-y+\frac{2}{5}z=-1$ — (2)

$\frac{12}{5}z=12$

$\Rightarrow z = 12 \times \frac{5}{12} = 5 \Rightarrow \boxed{z=5}$

put $z=5$ in (2), $-y+2=-1 \Rightarrow -y=-3$

$\Rightarrow \boxed{y=3}$

put $z=5, y=3$ in (1), $x+3+5=9$

$\Rightarrow x = 9-8 = 1$

(4) $\boxed{x=1}$

(2) Solve the following equations by Gauss elimination method.

Sol: $2x+y+4z=12,\ 8x-3y+2z=20$

$4x+11y-z=33$

Sol:

The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 33 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 9 & -9 & 9 \end{array}\right] \quad R_2 \to \frac{R_2}{-7}$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & 2 & 4 \\ 0 & 9 & -9 & 9 \end{array}\right] \quad R_3 \to R_3 - 9R_2$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -27 & -27 \end{array}\right]$$

From the last step.

$2x+y+4z=12$ — (1)

$y+2z=4$ — (2)

$-27z=-27$

$\Rightarrow \boxed{z=1}$

put $z=1$ in (2) $y+2(1)=4 \Rightarrow y=4-2=2$

put $z=1, y=2$ in (1), $2x+2+4=12$

$\Rightarrow 2x = 12-6 = 6 \Rightarrow 2x=6 \Rightarrow x=3.$

Gauss - Jordan Method.

1 Solve the following system of equations by Gauss Jordan Method.

$5x-2y+3z=18,\; x+7y-3z=-22,\; 2x-y+6z=22.$

Sol:

The augmented matrix is

$$[A,B] = \left[\begin{array}{ccc|c} 5 & -2 & 3 & 18 \\ 1 & 7 & -3 & -22 \\ 2 & -1 & 6 & 22 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -3 & -22 \\ 5 & -2 & 3 & 18 \\ 2 & -1 & 6 & 22 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -3 & -22 \\ 0 & -37 & 18 & 128 \\ 0 & -15 & 12 & 66 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 7 & -3 & -22 \\ 0 & 1 & \frac{-18}{37} & \frac{-128}{37} \\ 0 & -15 & 12 & 66 \end{array}\right] \quad R_2 \to \frac{R_2}{-37}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{15}{37} & \frac{82}{37} \\ 0 & 1 & \frac{-18}{37} & \frac{-128}{37} \\ 0 & 0 & \frac{174}{37} & \frac{522}{37} \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - 7R_2 \\ R_3 \to R_3 + 15R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc} 1 & 0 & \frac{15}{37} & \frac{82}{37} \\ 0 & 1 & -\frac{18}{37} & \frac{-128}{37} \\ 0 & 0 & 1 & 3 \end{array}\right] R_3 \to \frac{37}{174} R_3 \quad \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_1 \to R_1 - \frac{15}{37} R_3 \\ R_2 \to R_2 + \frac{18}{37} R_3 \end{array}$$

$\therefore x=1,\; y=-2,\; z=3$

2 Solve the following system of equations by Gauss Jordan method.

$x+y+z=9,\; 2x-3y+4z=13,\; 3x+4y+5z=40.$

Sol:

The augmented matrix

$$[A,B] = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & -\frac{2}{5} & 1 \\ 0 & 1 & 2 & 13 \end{bmatrix} \quad R_2 \Rightarrow \frac{R_2}{-5}$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{7}{5} & 8 \\ 0 & 1 & \frac{-2}{5} & 1 \\ 0 & 0 & \frac{12}{5} & 12 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 7/5 & 8 \\ 0 & 1 & \frac{-2}{5} & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad R_3 \to \frac{5}{12} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 - \frac{7}{5} R_3 \\ R_2 \to R_2 + \frac{2}{5} R_3 \end{array}$$

$\therefore x=1,\; y=3,\; z=5.$

Gauss - Jordan Method.

* order of convergence of Newton's method is 2.

① Compute the real root of $x \log_{10} x = 1.2$ correct to three decimal places using Newtons Method.

Sol:

Take $f(x) = x \log_{10} x - 1.2$

$f'(x) = \log_{10} x + \cancel{x} \frac{1}{\cancel{x}} \log_{10} e$ $\left[\because \frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e\right]$

$f(0) = -ve$, $f(1) = -ve$

$f(2) = 2\log_{10} 2 - 1.2 = 2(0.3010) - 1.2 = -0.5980$

$f(3) = 3\log_{10} 3 - 1.2 = 3(0.3010) - 1.2 = 0.2313$

Here $f(2)$ and $f(3)$ are opposite in sign

$\therefore$ the root lies between '2' and '3'

and $|f(2)| > |f(3)|$

$\boxed{\log_{10} e = 0.4343}$

$\therefore$ Take $x_0 = 2.7$

$f(x_0) = f(2.7) = -0.035$

$f'(x_0) = f'(2.7) = \log_{10} 2.7 + \log_{10} e = 0.867$

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.7 - \left(\frac{-0.035}{0.867}\right) = 2.740$

$f(x_1) = f(2.740) = -0.0006$

$f'(x_1) = f'(2.740) = 0.872$

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.740 - \frac{(-0.0006)}{0.872}$

$\boxed{x_2 = 2.741}$, $f(x_2) = f(2.741) = 0.003$

$f'(x_2) = f'(2.741) = 0.872$,

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.741 - \frac{0.003}{0.872}$

$\boxed{x_3 = 2.741}$

$\therefore$ The required root is 2.741.

② Find the real root of $xe^x - 2 = 0$ correct to three places of decimals using newtons method.

Soln:

$f(x) = xe^x - 2$, $f'(x) = xe^x + e^x = e^x(1+x)$

$f(0) = -2$, $f(1) = e - 2 = 2.7183 - 2$
(-ve) $= 0.7183$ (+ve)

$\therefore$ The root is lies between '0' and '1'

and $|f(0)| > |f(1)|$

$\therefore$ we take $x_0 = 1$

$f(x_0) = f(1) = 0.7183$,

$f'(x_0) = f'(1) = e(2) = 5.4366$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{0.7183}{5.4366} = 0.8679$

$f(x_1) = f(0.8679) = 0.0673$.

$f'(x_1) = f'(0.8679) = 4.4492$.

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.8679 - \frac{0.0673}{4.4492}$

$= 0.8679 - 0.0151 = 0.8528$ $\boxed{x_2 = 0.8528}$

$f(x_2) = f(0.8528) = .0008$

$f'(x_2) = f'(0.8528) = 4.3471$

$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.8528 - \frac{0.0008}{4.3471}$ $\boxed{x_3 = 0.8526}$

Since x_2 and x_3 are approximatly equal

$\therefore$ We take root is $\boxed{0.853}$

(3) Find the real root of x^3-3x+1 lying between '1' and '2' upto three decimal places by Newton's – Raphson method.

Sol.

$f(x) = x^3-3x+1, \quad f'(x) = 3x^2-3$

$f(1) = -1, \quad f(2) = 3$ and $|f(1)| < |f(2)|$

$\therefore$ Take $x_0 = 1$

$f(x_0) = f(1) = -1, \quad f'(x_0) = f'(1) = 0$

$\therefore$ we take $x_0 = \frac{1+2}{2} = 1.5$.

$f(x_0) = f(1.5) = -0.125, \quad f'(x_0) = 3.75$

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.5 - \frac{(-0.125)}{3.75}$

$$\boxed{x_1 = 1.5333}$$

$f(x_1) = f(1.5333) = 0.0049$

$f'(x_1) = 4.053$,

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5333 - \frac{0.0049}{4.053}$

(ie) $\boxed{x_2 = 1.5321}$

Now $f(x_2) = 3.5963 - 4.5963 + 1$
$= 0$

$\therefore \boxed{x_2 = 1.5321}$ is the required root of $f(x)$.

Condition for convergence of Newton's method
$|f''(x)\, f(x)| < |f'(x)|^2$

(4) Find an iterative formula to find $\sqrt{N}$ (N is +ve integer) and hence find $\sqrt{5}$.

Soln

Let $x = \sqrt{N} \Rightarrow x^2 = N$

$\Rightarrow x^2 - N = 0$.

Take $f(x) = x^2-N, \quad f'(x) = 2x$

Now $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$= x_n - \frac{(x_n^2-N)}{2x_n} = \frac{2x_n^2 - x_n^2 + N}{2x_n}$

(ie) $x_{n+1} = \frac{x_n^2+N}{2x_n} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$

$\therefore \boxed{x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right)}$ is an iterative formula to find $\sqrt{N}$ — (1)

Since $\sqrt{5}$ is lies between '2' and '3'

$\therefore$ we take $x_0 = 2$.

from (1) $x_1 = \frac{1}{2}\left(x_0 + \frac{5}{x_0}\right) = \frac{1}{2}\left(2 + 5/2\right) = 2.25$

$x_2 = \frac{1}{2}\left(x_1 + \frac{5}{x_1}\right) = \frac{1}{2}\left(2.25 + \frac{5}{2.25}\right) = 2.2361111$

$x_3 = \frac{1}{2}\left(2.2361111 + \frac{5}{2.2361111}\right) = 2.23606798$

$x_4 = \frac{1}{2}\left(2.23606798 + \frac{5}{2.23606798}\right) = 2.23606798$

$\therefore$ The root is $\boxed{2.23606798}$

Solve the following system of equation using - Gauss - Jordan's method.

Soln :-

Gn $5x_1 - x_2 = 9$, $-x_1 + 5x_2 - x_3 = 4$

$-x_2 + 5x_3 = -6$

$$[A, B] = \begin{bmatrix} 5 & -1 & 0 & 9 \\ -1 & 5 & -1 & 4 \\ 0 & -1 & 5 & -6 \end{bmatrix} \quad R_2 \Leftrightarrow 5R_2 + R_1$$

$$\sim \begin{bmatrix} 5 & -1 & 0 & 9 \\ 0 & 24 & -5 & 29 \\ 0 & -1 & 5 & -6 \end{bmatrix} \quad \begin{matrix} R_1 \Leftrightarrow 24R_1 + R_2 \\ R_3 \Leftrightarrow 24R_3 + R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 120 & 0 & -5 & 245 \\ 0 & 24 & -5 & 29 \\ 0 & 0 & 115 & -115 \end{bmatrix} \quad \begin{matrix} R_1 \Leftrightarrow R_1/5 \\ R_3 \Leftrightarrow \frac{R_3}{115} \end{matrix}$$

$$\sim \begin{bmatrix} 24 & 0 & -1 & 49 \\ 0 & 24 & -5 & 29 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \begin{matrix} R_1 \Leftrightarrow R_1 + R_3 \\ R_2 \Leftrightarrow R_2 + 5R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 24 & 0 & 0 & 48 \\ 0 & 24 & 0 & 24 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$\Rightarrow$ $x_1 = 2$

$x_2 = 1$

$x_3 = -1$

Eigen values and Eigen vectors

power method

Introduction:-

For each of these eigen values, the system of equations $(A-\lambda I)X = 0$ has a non trivial solution for the vector $x = (x_1, x_2, \dots x_n)^T$. This soln $x = (x_1, x_2 \dots x_n)^T$ is called a latent vector (or) Eigen vector corresponding to the eigen value λ.

power method:-

This method can be applied to find numerically the greatest eigen value of a square matrix (also called the dominant eigen value).

Let $\lambda_1, \lambda_2, \dots \lambda_n$ be the eigen values of A and let λ_1 be the dominant eigen value.

(ie) $|\lambda_1| \geq |\lambda_2| \geq \dots |\lambda_n|$

If the corresponding eigen vectors, are $x_0, x_1, \dots x_n$, then any arbitrary vector y can be written as

$y = a_0x_0 + a_1x_1 + \dots + a_nx_n$, since the eigen vectors are linearly independent. Now.

$A^k y = (a_0 x_0 + a_1 x_1 + \cdots + a_n x_n) A^k$

$= a_0 \lambda_1^k x_0 + a_1 x_1 \lambda_2^k + \cdots + a_n \lambda_n^k x_n$

$= \lambda_1^k \left(a_0 x_0 + a_1 x_1 \left(\frac{\lambda_2}{\lambda_1}\right)^k + \cdots \right)$

But $\left| \frac{\lambda_i}{\lambda_1} \right| < 1$. $(i = 1, 2 \ldots n)$

Hence $A^k y = \lambda_1^k a_0 x_0$

and $A^{k+1} y = \lambda_1^{k+1} a_0 x_0$

Hence, if k is large, $\lambda_1 = \frac{A^{k+1} y}{A^k y}$

where division is carried out in the corresponding components.

note:-

(1) If $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of A, then the eigen values λ_1 is dominant if $|\lambda_1| > |\lambda_i|$ for.
The eigen vector $X = (x_1, x_2 \ldots x_n)^T$ corresponding to the eigen values λ_1 is called the dominant eigen vector.

2. If the eigen values A are $-3, 1, 2$ then -3 is dominant.

3. If the eigen values of A are $4, 1, -4$ then A has no dominant eigen values since $|-4| = 4$.

4. The power method works satisfactorily only if A has dominant eigen value.

Find numerically the numerically the largest eigen values and the corresponding eigen vector of

$$A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$$

Soln

Let $x_0 = (1\ 0\ 0)^T$ be the initial vector. Then.

$$A x_0 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix} = 25\begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix}$$

$$A x_1 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1.00 \\ 0.04 \\ 0.08 \end{bmatrix} = \begin{bmatrix} 25.2 \\ 1.12 \\ 1.68 \end{bmatrix} = 25.2\begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix}$$

$$A x_2 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix} = \begin{bmatrix} 25.18 \\ 1.12 \\ 1.72 \end{bmatrix} = 25.18\begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix}$$

$$A x_3 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix} = \begin{bmatrix} 25.18 \\ 1.12 \\ 1.72 \end{bmatrix} = 25.18\begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix}$$

$\therefore$ The largest eigen values is 25.18. and the corresponding eigen vector is $(1, 0.4, 0.07)^T$.

Find the dominant eigen value and the corresponding eigen vector of $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and also the other two eigen values.

<u>Soln</u> :- we choose the initial vector $X_0 = (1\ 0\ 0)^T$

$$AX_0 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$AX_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 0.429 \\ 0 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.429 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.574 \\ 1.858 \\ 0 \end{bmatrix} = 3.574\begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} = 4.12\begin{bmatrix} 1 \\ 0.495 \\ 0 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.495 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.97 \\ 1.99 \\ 0 \end{bmatrix} = 3.97\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}.$$

$$AX_6 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

$\therefore$ The largest eigen value $A = 4$ and its eigen vector is $(1, 0.5, 0)^T$.

The least eigen value of A is the largest eigen values of $B = A - 4I$.

$$B = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

By power method, the dominant eigen value of B is obtained as follows.

Let $x_0 = (1\ 0\ 0)^T$ be the initial vector,

$$Bx_0 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -0.333 \\ 0 \end{bmatrix}$$

$$Bx_1 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.666 \\ 0 \end{bmatrix} = (-5)\begin{bmatrix} 1 \\ -0.333 \\ 0 \end{bmatrix}$$

$$Bx_2 = \begin{bmatrix} 3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.666 \\ 0 \end{bmatrix} = -5\begin{bmatrix} 1 \\ -0.333 \\ 0 \end{bmatrix}$$

$\therefore$ Dominant eigen value of $B = -5$

$\therefore$ The smallest eigen value of A is $-5+4 = -1$

By property,

Sum of the eigen values = Trace of A

$$\lambda_1 + 4 - 1 = 1 + 2 + 3$$

$$\lambda_1 + 3 = 6 \Rightarrow \lambda_1 = 3$$

The eigen values of A are $-1, 4, 3$.

Inverse of a matrix by Gauss - Jordan method

Introduction:

A square matrix whose determinant value is not zero is called a non singular matrix. Every non singular matrix has an inverse matrix. In this topic we shall find the inverse of the non singular square matrix A of order three. If X is the inverse of A then AX = I, where I is the identity matrix of same order.

By Gauss - Jordan method, the inverse matrix X is obtained by the following steps.

Step 1. First consider augmented matrix [A, I]

Step 2 Reduce the matrix A in [A I] to the identity matrix I by employing row transformations.

The row transformations used in step 2 transform I to A^{-1}

Finally write the inverse matrix A^{-1}. So the principle involved for finding A^{-1} shown below.

$$[A/I] \longrightarrow [I/A^{-1}]$$

Find the inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ by Gauss - Jordan method.

Soln

Consider

$$[A, I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] R_1 \to R_1 - 3R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

Thus $[A / I] \longrightarrow [I / A^{-1}]$

Hence $A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Find the inverse of $A = \begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix}$ using Gauss – Jordan method.

Soln

$$[A, I] = \left[\begin{array}{ccc|ccc} 2 & -2 & 4 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 1 & -1 & 0 & 0 & 1 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 2 & -2 & 4 & 1 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 5 & 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 & 0 & 2 \end{array}\right] \quad R_2 \to \frac{1}{5} R_2$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 2 & 1 & 0 & 2 \end{array}\right] \quad R_1 \to R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 0 & -1 & 0 & -\frac{1}{5} & \frac{3}{5} \\ 0 & 1 & 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 2 & 1 & 0 & 2 \end{array}\right] \quad R_3 \to \frac{R_3}{2}$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 0 & -1 & 0 & -\frac{1}{5} & \frac{3}{5} \\ 0 & 1 & 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 1 \end{array}\right] \quad R_1 \to R_1 + R_3$$

$$\sim \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & \frac{1}{2} & -\frac{1}{5} & \frac{8}{5} \\ 0 & 1 & 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 1 \end{array}\right] \quad R_1 \to \frac{R_1}{(-1)}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{5} & -\frac{8}{5} \\ 0 & 1 & 0 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{5} & -8/5 \\ 0 & \frac{1}{5} & 2/5 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$$

Iterative method

These methods are used to solve a special system of linear equations in which each equation must possess one large co-efficient and large large co-efficient must be attached to a different unknown in that equation. Further in each equation, the absolute of the large co-efficient of the unknown is greater than the sum of the absolute values of the other co-efficients of the other unknowns. Such type of simultaneous linear equations can be solved by the following iterative methods

(i) Gauss - Jacobi method

(ii) Gauss - Seidel method.

Gauss - Seidel method

consider the system equation

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Let us assume

$$|a_1| > |b_1| + |c_1|$$

$$|b_2| > |c_2| + |a_2|$$

$$|c_3| > |a_3| + |b_3|$$

(a) The co-efficient matrix

$$A = \begin{bmatrix} (a_1) & b_1 & c_1 \\ a_2 & (b_2) & c_2 \\ a_3 & b_3 & (c_3) \end{bmatrix}$$ is diagonally dominant.

Solving the given system for x, y, z, we have

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

In this method, once a new value for a unknown is found, it is used immediately for computing the new values of the unknown. We start with initial values $x^{(0)}, y^{(0)}, z^{(0)}$, the first iteration values are.

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1 y^{(0)} - c_1 z^{(0)})$$

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_1 z^{(0)})$$

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - c_3 z^{(1)})$$

If $x^{(r)}, y^{(r)}, z^{(r)}$ are the r^{th} iteration values, the iteration scheme for G.S method will be

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1 y^{(r)} - c_1 z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2 x^{(r+1)} - c_2 z^{(r)})$$

$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3 x^{(r+1)} - c_3 z^{(r+1)})$$

Hence in finding the values of the unknowns, we use the latest available values on the RHS.
The process of iteration is continued until the convergence is obtained with desired accuracy.

Rate of convergence

The rate of convergence of Gauss - Seidel method is roughly two times that of Gauss - Jacobi method

Further convergence Gauss-Seidel method is very fast than in Gauss - Jacobi, since the current values of the unknown are used immediately in each stage of iteration for getting the values of the unknowns.

Solve the following system of equation using - Gauss - Seidel method.

$8x - 3y + 2z = 20$, $4x + 11y - z = 33$
$6x + 3y + 12z = 35$

Soln

Here the co-efficient matrix

$$A = \begin{bmatrix} 8 & -3 & 2 \\ 4 & 11 & -1 \\ 6 & 3 & 12 \end{bmatrix}$$ is diagonally dominant

Solving for x, y, z, we have

$$x = \frac{1}{8}[3y - 2z + 20]$$

$$y = \frac{1}{11}[-4x + z + 33]$$

$$z = \frac{1}{12}[35 - 6x - 3y]$$

First iteration

$x = \frac{1}{8}[20 + 0 + 0] = 2.5$

$y = \frac{1}{11}[33 - 4(2.5) + 0] = 2.091$

$z = \frac{1}{12}[35 - 6(2.5) - 3(2.091)] = 1.144$

Second iteration

$x = \frac{1}{8}[20 + 3(2.091) - 2(1.144)] = 2.998$

$y = \frac{1}{11}[33 - 4(2.998) + 1.144] = 2.014$

$z = \frac{1}{12}[35 - 6(2.998) - 3(2.014)] = 0.914$

Third iteration

$x = \frac{1}{8}[20 + 3(2.998) - 2(0.914)] = 3.027$

$y = \frac{1}{11}[33 - 4(3.027) + 0.914] = 1.982$

$z = \frac{1}{12}[35 - 6(3.027) - 3(1.982)] = 0.908$

Fourth iteration

$x = \frac{1}{8}[20 + 3(3.027) - 2(1.982)] = 3.016$

$y = \frac{1}{11}[33 - 4(3.016) + 0.908] = 1.986$

$z = \frac{1}{12}[35 - 6(3.016) - 3(1.986)] = 0.912$

Fifth iteration

$x = \frac{1}{8}[20 + 3(3.016) - 2(1.986)] = 3.017$

$y = \frac{1}{11}[33 - 4(3.017) + 0.912] = 1.986$

$z = \frac{1}{12}[35 - 6(3.017) - 3(1.986)] = 0.912$

Sixth iteration.

$x = \frac{1}{8}[20 + 3(3.017) - 2(0.986)] = 3.017$

$y = \frac{1}{11}[33 - 4(3.017) + 0.912] = 1.986$

$z = \frac{1}{12}[35 - 6(3.017) - 3(1.986)] = 0.912$

$\therefore$ The solution is

$x = 3.017,\ y = 1.986,\ z = 0.912$

Solve the following equations.

$10x + 2y + z = 9$

$2x + 30y - 2z = -44$

$-2x + 3y + 10z = 22$

by Gauss - Seidel method.

Soln

The co-efficient matrix of the given system is diagonally dominant.

Solving for x, y, z, we have

$x = \frac{1}{10}[9 - 2y - z]$

$y = \frac{1}{30}[-44 - 2x + 2z]$

$z = \frac{1}{10}[22 + 2x - 3y]$

We start with initial value

$x = y = z = 0$

the iterated values are tabulated as follows.

Iteration	x	y	z
1	0.9	-1.527	2.838
2	0.922	-1.339	2.786
3	0.889	-1.340	2.779
4	0.89	-1.341	2.780
5	0.89	-1.341	2.780

∴ The soln is $x = 0.89$, $y = -1.341$

$z = 2.780$.

Solve the following system of equations by Gauss – Seidel method.

$3x - 13y - 3z = 49$, $5x - 6y + 17z = 45$

$11x + 2y - 2z = -31$

Solution

Rearranging the given system, we have

$$11x + 2y - 2z = -31$$
$$3x - 13y - 3z = 49$$
$$5x - 6y + 17z = 45$$

∴ The co-efficient matrix of the rearranged system is diagonally dominant.

Solving for x, y, z, we get

$$x = \frac{1}{11}\left[-31 - 2y + 2z\right]$$
$$y = -\frac{1}{13}\left[49 - 3x + 3z\right]$$
$$z = \frac{1}{17}\left[45 - 5x + 6y\right]$$

We start with initial values $x = y = z = 0$

The iteration values are tabulated as follows

Iteration	x	y	z
1	-2.818	-4.419	1.916
2	-1.666	-4.596	1.575
3	-1.707	-4.513	1.556
4	-1.713	-4.524	1.555
5	-1.713	-4.523	1.555
6	-1.713	-4.523	1.555

∴ The solution is $x = -1.713$

$y = -4.523$, $z = 1.555$.