**Solution to the homework 3 GTS211**

1. Find and use a special integrating factor to solve

(a) $(x^4 - x + y)dx - xdy = 0$

**Solution**

First, check the exactness of the equation, let $M = x^4 - x + y$, $N = -x$

$$\frac{\partial M}{\partial y} = 1, \frac{\partial N}{\partial x} = -1$$

The equation is non-exact.

Find the integrating factor,

Suppose we have integrating factor $f(x)$,

$$f(x) = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx} = e^{\int \frac{-1}{x}(1-(-1))dx} = x^{-2}$$

Multiply both sides by integrating factor,

$$x^{-2}(x^4 - x + y)dx - \frac{1}{x}dy = 0$$

Since the equation is exact, we have

$F(x,y) = \int x^{-2}(x^4 - x + y)dx + g(y)$

$F(x,y) = \int x^2 - \frac{1}{x} + \frac{y}{x^2}dx + g(y)$

$F(x,y) = \frac{x^3}{3} - \ln(x) - \frac{y}{x} + g(y)$

$\frac{\partial F}{\partial y} = \frac{-1}{x} + g'(y) = \frac{-1}{x}$

$g'(y) = 0$

$g(y) = \int 0 \ dy + c$

$g(y) = c$

The equation becomes

$\frac{x^3}{3} - \ln(x) - \frac{y}{x} - c = 0$

Rearrange the equation,

$y = \frac{x^4}{3} - xln(x) - cx$

(b) $(2xy)dx + (y^2 - 3x^2)dy = 0$

**Solution**

First, check the exactness of the equation, let $M = 2xy$, $N = y^2 - 3x^2$

$$\frac{\partial M}{\partial y} = 2x, \frac{\partial N}{\partial x} = -6x$$

The equation is non-exact.

Find the integrating factor,

Suppose we have integrating factor $f(x)$,

$$f(x) = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx} = e^{\int \frac{1}{y^2 - 3x^2}(2x - (-6x))dx}$$

The integrating factor $f(x)$ does not work. Let integrating factor to be $f(y)$,

$$f(y) = e^{\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dy} = e^{\int \frac{1}{2xy}(-6x - 2x)dy} = y^{-4}$$

Multiply both sides by integrating factor,

$$y^{-4}(2xy)dx + y^{-4}(y^2 - 3x^2)dy = 0$$

Since the equation is exact, we have

$F(x, y) = \int y^{-4}(2xy)dx + g(y)$

$F(x, y) = \int \frac{2x}{y^3} dx + g(y)$

$F(x, y) = \frac{x^2}{y^3} + g(y)$

$\frac{\partial F}{\partial y} = \frac{-3x^2}{y^4} + g'(y) = \frac{y^2 - 3x^2}{y^4}$

$g'(y) = \frac{1}{y^2}$

$g(y) = \int \frac{1}{y^2}\, dy + c$

$g(y) = \frac{-1}{y} + c$

The equation becomes

$\frac{x^2}{y^3} - \frac{1}{y} + c = 0$

2.

(a) $(xy + y^2)dx - x^2 dy = 0$

**Solution**

Rearrange the equation,

$$\frac{dy}{dx} = \frac{xy + y^2}{x^2} = \frac{y}{x} + \frac{y^2}{x^2}$$

Let $u = \frac{y}{x}, xu = y$

Take the derivative respect to $x$ both sides,

$u + x\frac{du}{dx} = \frac{dy}{dx}$ , substitute this into the equation,

$$u + x\frac{du}{dx} = u + u^2$$

$$x\frac{du}{dx} = u^2$$

$$\frac{1}{u^2}du = \frac{1}{x}dx$$

Integrate both sides,

$$\int \frac{1}{u^2}du = \int \frac{1}{x}dx + C$$

$$-\frac{1}{u} = \ln(x) + C$$

$$\frac{-x}{y} = \ln(x) + C$$

$$y = \frac{-x}{\ln(x) + C}$$

(b) $(y^2 - xy)dx + x^2 dy = 0$

**Solution**

Rearrange the equation,

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2} = \frac{y}{x} - \frac{y^2}{x^2}$$

Same solution as (a), at last we have,

$$u + x\frac{du}{dx} = u - u^2$$

$$x\frac{du}{dx} = -u^2$$

$$\frac{-1}{u^2}du = \frac{1}{x}dx$$

Integrate both sides,

$$\int \frac{-1}{u^2}du = \int \frac{1}{x}dx + C$$

$$\frac{1}{u} = \ln(x) + C$$

$$\frac{x}{y} = \ln(x) + C$$

$$y = \frac{x}{\ln(x) + C}$$

(c) $\frac{dy}{dx} = (x - 5y + 2)^2$

**<u>Solution</u>**

Let $z = x - 5y + 2$, and take derivative respect to $x$

$$\frac{dz}{dx} = 1 - 5\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1 - \frac{dz}{dx}}{5}$$

Substitute into the question,

$$\frac{1 - \frac{dz}{dx}}{5} = z^2$$

$$\frac{dz}{dx} = 1 - 5z^2$$

$$\frac{dz}{1 - 5z^2} = dx$$

Integrate both sides,

$$\int \frac{dz}{1 - 5z^2} = \int dx + C$$

$$\int \frac{dz}{1 - 5z^2} = x + C$$

Find $\int \frac{dz}{1-5z^2}$

Let $z = \frac{1}{\sqrt{5}}\sin\theta$, $dz = \frac{1}{\sqrt{5}}\cos\theta\, d\theta$

$$\int \frac{dz}{1 - 5z^2} = \int \frac{\frac{1}{\sqrt{5}}\cos\theta\, d\theta}{1 - 5(\frac{1}{\sqrt{5}}\sin\theta)^2} = \int \frac{1}{\sqrt{5}}\frac{\cos\theta\, d\theta}{1 - sin^2\theta} = \int \frac{1}{\sqrt{5}}\frac{\cos\theta\, d\theta}{(\cos\theta)^2} = \int \frac{1}{\sqrt{5}} sec\theta d\theta$$

$$= \frac{1}{\sqrt{5}}\ln(sec\theta + tan\theta) = \frac{1}{\sqrt{5}}\ln(sec\theta + tan\theta)$$

From $\sin\theta = \sqrt{5}z$, construct the right triangle, we have $\tan\theta = \frac{\sqrt{5}z}{\sqrt{1-5z^2}}, sec\theta = \frac{1}{\sqrt{1-5z^2}}$

$$\int \frac{dz}{1-5z^2} = \frac{1}{\sqrt{5}}\ln\left(\frac{1+\sqrt{5}z}{\sqrt{1-5z^2}}\right)$$

$$\frac{1}{\sqrt{5}}\ln\left(\frac{1+\sqrt{5}z}{\sqrt{1-5z^2}}\right) = x + C$$

$$\frac{1}{\sqrt{5}}\ln\left(\frac{1+\sqrt{5}(x-5y+2)}{\sqrt{1-5(x-5y+2)^2}}\right) = x + C$$

After the very huge rearranging the term, we have

$$y = \frac{1}{5\sqrt{5}}\left(\frac{1-e^{2\sqrt{5}x+2C}}{1+e^{2\sqrt{5}x+2C}}\right) + \frac{x}{5} + \frac{2}{5}$$

$$y = -\frac{1}{5\sqrt{5}}\tanh\left(\sqrt{5}x + C\right) + \frac{x}{5} + \frac{2}{5}$$

(d) $\frac{dy}{dx} = \sin(x-y)$

**Solution**

Let $v = x - y, \frac{dv}{dx} = 1 - \frac{dy}{dx}, \frac{dy}{dx} = 1 - \frac{dv}{dx},$

$$1 - \frac{dv}{dx} = \sin v$$

$$\frac{dv}{dx} = 1 - \sin v$$

$$\frac{dv}{1-\sin v} = dx$$

$$\int \frac{dv}{1-\sin v} = \int dx + C$$

$$\int \frac{dv}{1-\sin v} = x + C$$

Find $\int \frac{dv}{1-\sin v}$

$$\int \frac{dv}{1-\sin v} = \int \left(\frac{1}{1-\sin v}\right)\left(\frac{1+\sin v}{1+\sin v}\right)dv = \int \frac{1+\sin v}{1-sin^2 v}dv = \int \frac{1+\sin v}{cos^2 v}dv$$

$$= \int \frac{1}{cos^2 v} + \frac{\sin v}{cos^2 v}dv = \int sec^2 v\, dv + \int \sec(v)\tan(v)\, dv = \tan(v) + \sec(v)$$

$$\tan(v) + \sec(v) = x + C$$

$$\frac{\sin(x-y)+1}{\cos\,(\mathrm{x}-\mathrm{y})} = x + C$$

(e) $\frac{dy}{dx} - y = \; e^{2x}y^3$

**Solution**

Divide both sides by $y^3$

$$\frac{1}{y^3}\frac{dy}{dx} - \frac{1}{y^2} = e^{2x}$$

Let $w = \; \frac{1}{y^2}, \frac{dw}{dx} = \; \frac{-2}{y^3}\frac{dy}{dx}\; , \frac{1}{y^3}\; \frac{dy}{dx} = \frac{-1}{2}\,(\frac{dw}{dx})$

$$\frac{-1}{2}\frac{dw}{dx} - w = e^{2x}$$

$$\frac{dw}{dx} + 2w = \; e^{2x}$$

$$(e^{2x} - 2w)dx - dw = 0$$

First, check the exactness of the equation, let $M = \; e^{2x} - 2w$, $N = -1$

$$\frac{\partial M}{\partial w} = \; -2, \frac{\partial N}{\partial x} = \; 0$$

The equation is non-exact.

Find the integrating factor,

Suppose we have integrating factor $f(x)$,

$$f(x) = \; e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial w} - \frac{\partial N}{\partial x}\right)dx} = e^{\int \frac{1}{-1}(-2-0)dx} = \; e^{2x}$$

Multiply both sides by integrating factor,

$$e^{2x}(e^{2x} - 2w)dx - e^{2x}dw = 0$$

Since the equation is exact, we have

$F(x,w) = \; \int e^{2x}(e^{2x} - 2w)dx + g(w)$

$F(x,w) = \; \int e^{4x} - 2we^{2x}dx + g(w)$

$F(x,w) = \; \frac{e^{4x}}{4} - we^{2x} + g(w)$

$\frac{\partial F}{\partial w} = \; -e^{2x} + g'(y) = \; -e^{2x}$

$g'(y) = \; 0$

$g(y) = \; \int 0 \; dy + c$

$g(y) = c$

The equation becomes

$\frac{e^{4x}}{4} - we^{2x} + C = 0$

Substitute the value of $y$

$$\frac{e^{4x}}{4} - \frac{e^{2x}}{y^2} + C = 0$$

$$y^2 = \frac{e^{2x}}{\frac{e^{4x}}{4} + C}$$

(f) $\frac{dy}{dx} + \frac{y}{x-2} = 5(x-2)y^{\frac{1}{2}}$

**Solution**

Let $t = \sqrt{y}, \frac{dt}{dx} = \frac{1}{2\sqrt{y}}\frac{dy}{dx}$

From the question, rearrange by dividing both sides by $y^{\frac{1}{2}}$,

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} + \frac{\sqrt{y}}{x-2} = 5(x-2)$$

$$2\frac{dt}{dx} + \frac{t}{x-2} = 5(x-2)$$

$$(5(x-2)^2 - t)dx - 2(x-2)dt = 0$$

First, check the exactness of the equation, let $M = 5(x-2)^2 - t, N = -2(x-2)$

$$\frac{\partial M}{\partial t} = -1, \frac{\partial N}{\partial x} = -2$$

The equation is non-exact.

Find the integrating factor,

Suppose we have integrating factor $f(x)$,

$$f(x) = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial t} - \frac{\partial N}{\partial x}\right)dx} = e^{\int \frac{1}{-2(x-2)}(-1-(-2))dx} = e^{\int \frac{1}{4-2x}dx} = \frac{1}{\sqrt{4-2x}}$$

Multiply both sides by integrating factor,

$$\frac{1}{\sqrt{4-2x}}(5(x-2)^2 - t)dx - \frac{2}{\sqrt{4-2x}}(x-2)dt = 0$$

Since the equation is exact, we have

$F(x,t) = \int -\frac{2}{\sqrt{4-2x}}(x-2)dt + g(x)$

$F(x,t) = \frac{2t}{\sqrt{4-2x}}(2-x) + g(x) = t\sqrt{4-2x}$

$\frac{\partial F}{\partial x} = \frac{-t}{\sqrt{4-2x}} + g'(x) = \frac{1}{\sqrt{4-2x}}(5(x-2)^2 - t)$

Multiply both sides by $\sqrt{4-2x}$

$$-t + g'(x) = (5(x-2)^2 - t)$$

$$g'(x) = 5(x-2)^2$$

$$g(x) = \int 5(x-2)^2 dx + C$$

$$g(x) = \frac{5}{3}(x-2)^2 + C$$

Hence, we have the equation,

$$t\sqrt{4-2x} + \frac{5}{3}(x-2)^2 + C = 0$$

$$\sqrt{y}\sqrt{4-2x} + \frac{5}{3}(x-2)^2 + C = 0$$

$$y = \frac{(\frac{5}{3}(x-2)^2 + C)^2}{4-2x}$$

(g) ,(h) have similar solutions as 1.(a) and 1.(b)

3.

**Solution**

$$\frac{dy}{dx} = 2\frac{y}{x} + \cos\left(\frac{y}{x^2}\right)$$

Let $y = vx^2, \frac{dy}{dx} = \frac{dv}{dx}x^2 + 2xv$

$$\frac{dv}{dx}x^2 + 2xv = 2xv + \cos(v)$$

$$\frac{dv}{\cos(v)} = \frac{dx}{x^2}$$

$$\int \sec(v)\, dv = \int \frac{dx}{x^2} + c$$

$$\ln(\sec(v) + \tan(v)) = -\frac{1}{x} + C$$

$$\ln\left(\frac{sin\left(\frac{y}{x^2}\right) + 1}{cos\left(\frac{y}{x^2}\right)}\right) = -\frac{1}{x} + C$$